Randomized Quicksort

Randomized Algorithms & Data Structures

- *Monte-Carlo algorithms*
  - Find the correct answer most of the time
  - Can usually amplify probability of success with repetitions
  - Example, Karger's min cut (in textbook)
- *Las-Vegas algorithms*
  - Always find the correct answer, e.g. RandQuick sort (today!)
  - But the worst-case running-time guarantees are not strong (they hold in expectation or with high probability, but their goodness depends on randomness)
- *Randomized data structures*: hashing, search trees, filters, etc.

Randomized Algorithm I

Randomized Selection

Randomized Selection

Problem. Find the kth smallest/largest element in an unsorted array

- Recall our selection algorithm from back in our divide and conquer unit (lecture 15):

Select (A, k):

If $|A| = 1$: return $A[1]$

Else:

Choose a pivot $p \leftarrow A[1, \ldots, n]$; let r be the rank of p

$r, A_{<p}, A_{>p} \leftarrow$ Partition((A, p))

If $k == r$: return p

Else if $k < r$: Select $(A_{<p}, k)$

Else: Select $(A_{>p}, k - r)$

Selection with a Good Pivot

- Recall: pivot is "good" if it reduced the array size by at least a constant
  - Gives a recurrence $T(n) \leq T(\alpha n) + O(n)$ for some constant $\alpha < 1$
  - Expands to a decreasing geometric series $T(n) = O(n)$
- In the deterministic algorithm, how did we find a good pivot?
  - Split array into groups of 5
  - And computed the median of group medians
  - The pivot guaranteed that $n \to 7n/10$
- **Here is a silly idea:** What if we pick the pivot uniformly at random?
  - Seems like the pivot is "usually" around the midpoint
  - What is the expected running time?

Randomized Selection

- **Problem.** Find the k^{th} smallest/largest element in an unsorted array
- Recall our selection algorithm

Select (A, k)**:**

If $|A| = 1$: return $A[1]$

Else:

Choose a pivot $p \leftarrow A[1,\ldots,n]$ uniformly at random; let r be the rank of p

$r, A_{<p}, A_{>p} \leftarrow$ Partition((A, p))

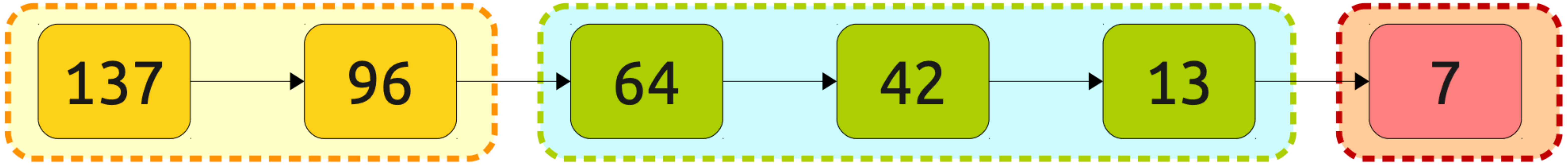
If $k == r$: return p

Else if $k < r$: Select $(A_{<p}, k)$

Else: Select $(A_{>p}, k - r)$

Analyzing Randomized Selection

- Normally, we'd write a recurrence relation for a recursive function
- A bit complicated now—input sizes of later recursive calls depend on the random choices of pivots in earlier calls
- We will use a different accounting trick for running time
- Randomized selection makes at most one recursive call each time:
  - Group multiple recursive call in "phases"
  - Sum of work done by all calls is equal to the sum of the work done in all the phases



Analyzing in Phases

- **Idea**: let a "phase" of the algorithm be the time it takes for the array size to drop by a constant factor (say $n \to (3/4) \cdot n$)
- If array shrinks by a constant factor in each phase and linear work is done in each phase, what would be the running time?
- $T(n) = c(n + 3n/4 + (3/4)^2 n + \ldots + 1) = O(n)$
- If we want a $1/4$th, $3/4$th split, what range should our pivot be in?
    - Middle half of the array (if n size array, then pivot in $[n/4, 3n/4]$)
    - What is the probability of picking such a pivot?
        - $1/2$
    - Phase ends as soon as we pick a pivot in the middle half
        - Expected # of recursive calls until phase ends? 2

Expected Running Time

- Let the algorithm be in phase j when the size of the array is
  - At least $n\left(\frac{3}{4}\right)^{j+1}$ but not greater that $n\left(\frac{3}{4}\right)^{j}$
- Expected number of iterations within a phase: 2
- Let X_j be the expected number of steps spent in phase j
- $X = X_0 + X_1 + X_2 \ldots$ be the total number of steps taken by the algorithm
- $\mathrm{E}(X_j) =$ $\mathrm{E}(\#$ recursive calls until jth phase ends $\cdot$ # steps in phase $j)$
- $\mathrm{E}(X_j) \leq cn(3/4)^j \cdot \mathrm{E}(\#$ recursive calls until jth phase ends) $= 2cn(3/4)^j$

Expected Running Time

- Let X_j be the expected number of steps spent in phase j
- $X = X_0 + X_1 + X_2 \ldots$ be the total number of steps taken by the algorithm
- $\mathrm{E}(X_j) = \mathrm{E}(\#$ of iterations until jth phase ends $\cdot$ # steps in phase $j)$
- $\mathrm{E}(X_j) \leq n(3/4)^j \cdot \mathrm{E}(\#$ iterations until jth phase ends$) = 2cn(3/4)^j$
- Now we can apply linearity of expectation:
- $$E[X] = \sum_j E[X_j] \leq \sum_j 2cn \left(\frac{3}{4}\right)^j = 2cn \sum_j \left(\frac{3}{4}\right)^j$$

$$= \Theta(n)$$

Pivot Selection

- Deterministic and random both take $O(n)$ time
    - What's the advantage of the deterministic algorithm?
    - Worst-case guarantee—the random algorithm could be very slow sometimes
    - What's the advantage of the random algorithm?
    - Much much simpler and better constants hidden in $O()$
- Which should you use?
    - Pretty much always random
    - Question to ask yourself:
        - how often is the randomized algorithm going to be much worse than $O(n)$?

Randomized Algorithm II

Randomized QuickSort

Randomized Quicksort

- Recall *deterministic* Quicksort
- Depending on the choice pivot, could be $O(n^2)$
- What if we pick the pivot uniformly at random?
    - We saw in randomized selection that this leads to good pivots half of the time

Quicksort(A)**:**

If $|A| < 3$: Sort(A) directly

Else: choose a pivot element $p \leftarrow A$

$A_{<p}, A_{>p} \leftarrow$ Partition around p

Quicksort$(A_{<p})$

Quicksort$(A_{>p})$

Randomized Quicksort

- Intuitively half the pivots will be good, half bad
- We will analyze quick sort using another accounting trick (see the textbook for example similar to selection's approach of analyzing "phases")
- Total work done can be split into to types:
  - Work done making recursive calls (this is a lower order term, it turns out)
  - Work partitioning the elements
- How many recursive calls in the worst case?
  - Imagine worst pivot being chosen each time
  - $O(n)$

Randomized Quicksort

- We thus need to bound the work partitioning elements
- Partitioning an array of size n around a pivot p takes exactly $n - 1$ comparisons
- We won't look at partitions made in each recursive call, which depend on the choice of random pivot
- **Idea:** Instead, account for the total work done by the partition step by summing up the total number of comparisons made
- Two ways to count total comparisons:
    - Look at the size of arrays across recursive calls and sum
    - Look at all pairs of elements and count total # of times they are compared (this is easier to do in this case)

Aside: Randomized Analysis

- Often multiple ways to determine a randomized algorithm’s cost
- We can split into phases, or count the cost directly. We can calculate each probability, or use linearity of expectation
- Intrinsically some “cleverness” involved in choosing the way that gets you a clean answer
- We’ll focus on problems where there’s a clear path to finding the solution (either it follows directly from the question, or we’ll revisit problems you’ve seen before). More complex problems abound if you look!
- That said, here’s a very clever way to calculate Quicksort’s running time

Counting Total Comparisons

- Just for analysis, let B denote the sorted version of input array A, that is, $B[i]$ is the ith smallest element in A
- Define random variable X_{ij} as the number of times Quicksort compares $B[i]$ and $B[j]$
- Observation: $X_{ij} = 0$ or $X_{ij} = 1$, why?
  - $B[i]$, $B[j]$ only compared when one of them is the current pivot; pivots are excluded from future recursive calls
- Let $T = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$ be the total number of comparisons made by randomized Quicksort

Expected Running Time

- **Goal**: $E[T] = E\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n}\sum_{j=i+1}^{n} E[X_{ij}]$
- $E[X_{ij}] = \Pr[X_{ij} = 1]$
- When is $X_{ij} = 1$? That is, when are $B[i]$ and $B[j]$ compared?
- Consider a particular recursive call. Let rank of pivot p be r.
    - Let's think about where $B[i], B[j]$ lie with respect to p

Expected Running Time

- Goal: $E[T] = E\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n}\sum_{j=i+1}^{n} E[X_{ij}]$
- $E[X_{ij}] = \Pr[X_{ij} = 1]$
- When is $X_{ij} = 1$? That is, when are $B[i]$ and $B[j]$ compared?
- Consider a particular recursive call. Let rank of pivot p be r.
  - Case 1. One of them is the pivot: $r = i$ or $r = j$
  - Case 2. Pivot is between them: $r > i$ and $r < j$
  - Case 3. Both less than the pivot: $r > i, j$
  - Case 4. Both greater than the pivot: $r < i, j$

Comparisons for Each Case

- **Case 1**. $r = i$ or $r = j$
  - $B[i]$ and $B[j]$ are compared once and one of them is excluded from all future calls
- **Case 2**. $r > i$ and $r < j$
  - $B[i]$ and $B[j]$ are both compared to the pivot but not to each other, after which they are in different recursive calls: will never be compared again
- **Case 3**. $r > i, j$ and **Case 4**. $r < i, j$
  - $B[i]$ and $B[j]$ are not compared to each other, they are both in the same subarray and may be compared in the future
- **Takeaway:** $B[i]$, $B[j]$ are compared for the 1st time when one of them is chosen as pivot from $B[i], B[i+1], \ldots, B[j]$ & never again

Expected Running Time

- Pr$[X_{ij} = 1] =$ Pr(one of them is picked as pivot from $B[i], B[i+1], \ldots, B[j]$
- Pr$[X_{ij} = 1] = \dfrac{2}{j - i + 1}$
- $E[T] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \dfrac{1}{j - i + 1}$

Expected Running Time

- $B[i]$ and $B[j]$ are compared iff one of them is the first pivot chosen from the range $B[i], B[i+1], \ldots, B[j]$
- $\Pr[X_{ij} = 1] = \dfrac{2}{j - i + 1}$

At each round, the probability that $X_{ij} = 1$ conditioned on the event that we are in Case 1 or Case 2.
(In Cases 3 and 4, $X_{ij} = 0$)

- $E[T] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \dfrac{1}{j - i + 1}$
- For fixed i, inner sum is $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots \dfrac{1}{n - i + 1} \leq \sum_{\ell=2}^{n} \dfrac{1}{\ell} = O(\log n)$
- Thus, expected number of comparisons is:
  $E[T] = O(n \log n)$

Quick Sort Summary

- Las Vegas algorithms like Quicksort and Selection are always correct and their running time guarantees hold ***in expectation***
- We can actually prove that the number of comparisons made by Quicksort is $O(n \log n)$ **with high probability**
  - W.H.P. means that the the probability that the running time of quicksort is more than a constant c factor away from its expectation is very small (polynomially small: less than $1/n^c$ for $c \geq 1$)
  - Whp bounds are called **concentration bounds**
  - Whp: ideal guarantees possible for a randomized algorithm

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  - Shikha Singh
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)