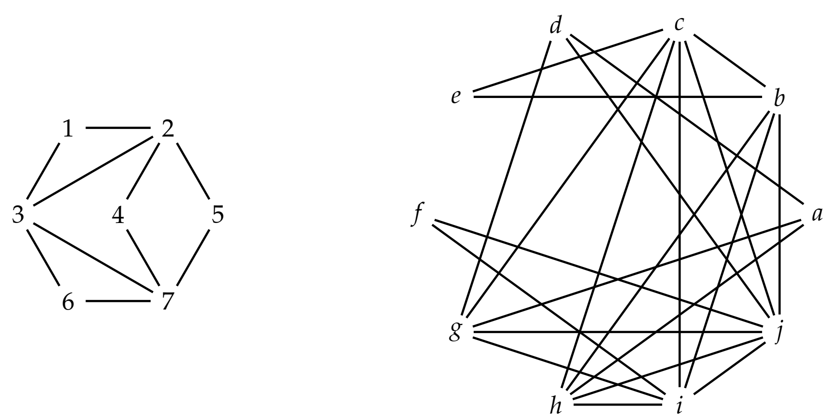
Algorithms: Introduction to reductions

Model 1: Independent sets

Definition 1. An *independent set* in an undirected graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that no two vertices in S are adjacent.

Definition 2. A *vertex cover* in an undirected graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that every edge $e \in E$ has at least one endpoint in (is "covered by") C.



Think of edges as hallways in an art museum and C as the set of locations where we are going to put some guards. Then an independent set means no two guards can see each other; a vertex cover means every hallway is watched by at least one guard.

1 Which of the following are independent sets?

(a) $\{1, 2\}$

(b) $\{1, 5\}$

(c) $\{c, a\}$

(d) $\{e, a, i, g\}$

(e) $\{7\}$

(f) $\varnothing$

2 For each graph, list at least three other examples of independent sets.

3 Given an arbitrary graph G, does G always have at least one independent set? Why or why not?

4 Intuitively, which is harder: to find big independent sets, or small ones? Why?

5 Based on the previous observation, an interesting question to ask about a given graph G is to find the _________________________.

6 Try to answer your interesting question for the given example graphs (but don't spend more than a few minutes). How sure are you about your answer?

7 Describe a brute-force algorithm to answer this question. What is its big-Θ running time in terms of $|V|$ and $|E|$?

8 Guess the running time (in terms of $|V|$ and $|E|$) of the fastest known algorithm to solve this problem. (You do not have to come up with an algorithm; just guess how fast you think this problem can be solved.)

9 Which of the following are vertex covers?

(a) $\{3, 4, 5, 6, 7\}$

(b) $\{2, 3, 4, 6, 7\}$

(c) $\{b, d, e, f, g, h, i, j\}$

(d) $\{b, c, d, f, h, j\}$

(e) $\{1, 2, 3, 4, 5, 6\}$

(f) $\{1, 2, 3, 4, 5, 6, 7\}$

10 For each graph, list at least three other examples of vertex covers.

11 Given an arbitrary graph G, does G always have at least one vertex cover? Why or why not?

12 Intuitively, which is harder: to find small vertex covers, or big ones? Why?

13 Based on the previous observation, an interesting question to ask about a given graph G is to find the ________________________.

14 Try to answer your interesting question for the given example graphs. How sure are you about your answer?

15 Describe a brute-force algorithm to answer this question. What is its big-Θ running time in terms of $|V|$ and $|E|$?

16 Compare your answers to questions 1 and 9. What do you notice?

Make a conjecture based on your observations in the previous section:

Theorem 3. *Let $G = (V, E)$ be an undirected graph, and $S \subseteq V$ a subset of its vertices. Then S is an independent set if and only if* ________________.

Let's prove it! This requires proving "both directions" of the claim. For the first direction, a skeleton proof is provided. For the reverse direction, you must write the proof from scratch.

Proof. ($\Longrightarrow$) Let S be an independent set. We must show

____________________. So pick an arbitrary edge $e = (u, v) \in E$;

by definition we must show that at least one of u or v __________,

that is, at least one of u or v is not ________________________.
Since S is an independent set and u and v are connected by an

edge, u and v can't both ________________________________,

and therefore ____.

Now, fill in the proof for the "other" direction!.

Write down what you get to assume and what you are trying to prove, and expand definitions.

($\Longleftarrow$) □