Model 1: Independent sets

**Definition 1.** An independent set in an undirected graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that no two vertices in $S$ are adjacent.

**Definition 2.** A vertex cover in an undirected graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that every edge $e \in E$ has at least one endpoint in (is “covered by”) $C$.

1. Which of the following are independent sets?
   
   (a) $\{1, 2\}$
   (b) $\{1, 5\}$
   (c) $\{c, a\}$
   (d) $\{e, a, i, g\}$
   (e) $\{7\}$
   (f) $\emptyset$

2. For each graph, list at least three other examples of independent sets.

3. Given an arbitrary graph $G$, does $G$ always have at least one independent set? Why or why not?
4. Intuitively, which is harder: to find big independent sets, or small ones? Why?

5. Based on the previous observation, an interesting question to ask about a given graph $G$ is to find the ________________.

6. Try to answer your interesting question for the given example graphs (but don’t spend more than a few minutes). How sure are you about your answer?

7. Describe a brute-force algorithm to answer this question. What is its big-$\Theta$ running time in terms of $|V|$ and $|E|$?

8. Guess the running time (in terms of $|V|$ and $|E|$) of the fastest known algorithm to solve this problem. (You do not have to come up with an algorithm; just guess how fast you think this problem can be solved.)

9. Which of the following are vertex covers?
   (a) $\{3, 4, 5, 6, 7\}$
   (b) $\{2, 3, 4, 6, 7\}$
   (c) $\{b, d, e, f, g, h, i, j\}$
   (d) $\{b, c, d, f, h, j\}$
   (e) $\{1, 2, 3, 4, 5, 6\}$
   (f) $\{1, 2, 3, 4, 5, 6, 7\}$

10. For each graph, list at least three other examples of vertex covers.
11 Given an arbitrary graph $G$, does $G$ always have at least one vertex cover? Why or why not?

12 Intuitively, which is harder: to find small vertex covers, or big ones? Why?

13 Based on the previous observation, an interesting question to ask about a given graph $G$ is to find the ____________.

14 Try to answer your interesting question for the given example graphs. How sure are you about your answer?

15 Describe a brute-force algorithm to answer this question. What is its big-$\Theta$ running time in terms of $|V|$ and $|E|$?

16 Compare your answers to questions 1 and 9. What do you notice?

Make a conjecture based on your observations in the previous section:

**Theorem 3.** Let $G = (V, E)$ be an undirected graph, and $S \subseteq V$ a subset of its vertices. Then $S$ is an independent set if and only if ______________.
Let’s prove it! This requires proving “both directions” of the claim. For the first direction, a skeleton proof is provided. For the reverse direction, you must write the proof from scratch.

**Proof.** \( \rightarrow \) Let \( S \) be an independent set. We must show

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\text{So pick an arbitrary edge } e = (u, v) \in E; \]

by definition we must show that at least one of \( u \) or \( v \) ________,

that is, at least one of \( u \) or \( v \) is not ____________________.

Since \( S \) is an independent set and \( u \) and \( v \) are connected by an edge, \( u \) and \( v \) can’t both ____________________,

and therefore ____________________.

Now, fill in the proof for the “other” direction! Write down what you get to assume and what you are trying to prove, and expand definitions.

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