Applications of Network Flow:

Solving Problems by Reduction to Network Flows

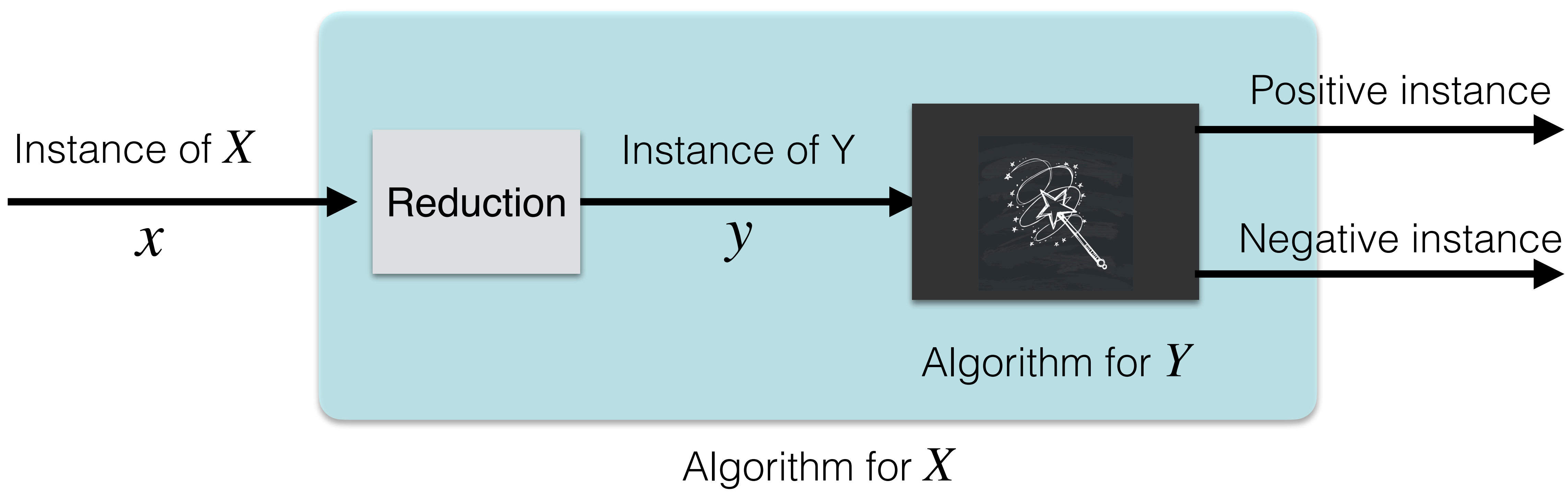
Reductions

- We will solve these problems by **reducing** them to a network flow problem
- We'll focus on the concept of **problem reductions**

Anatomy of Problem Reductions

At a high level, a problem X reduces to a problem Y if an algorithm for Y can be used to solve X

- **Reduction.** Convert an arbitrary instance x of X to a special instance y of Y such that there is a 1-1 correspondence between them



Anatomy of Problem Reductions

- **Claim.** x satisfies a property iff y satisfies a *corresponding* property
- Proving a reduction is correct: prove both directions
- x has a property (e.g. has matching of size k) $\implies y$ has a corresponding property (e.g. has a flow of value k)
- x does not have a property (e.g. does not have matching of size k) $\implies y$ does not have a corresponding property (e.g. does not have a flow of value k)
- Or equivalently (and this is often easier to prove):
  - y has a property (e.g. has flow of value k) $\implies x$ has a corresponding property (e.g. has a matching of value k)

Remaining Plan

We will explore one application of network flow in detail today

- Matching in bipartite graphs
- Matchings are super practical with many applications
- We have already seen one, can you remember?

Next meeting: another application reducible to network flow (playoff elimination)

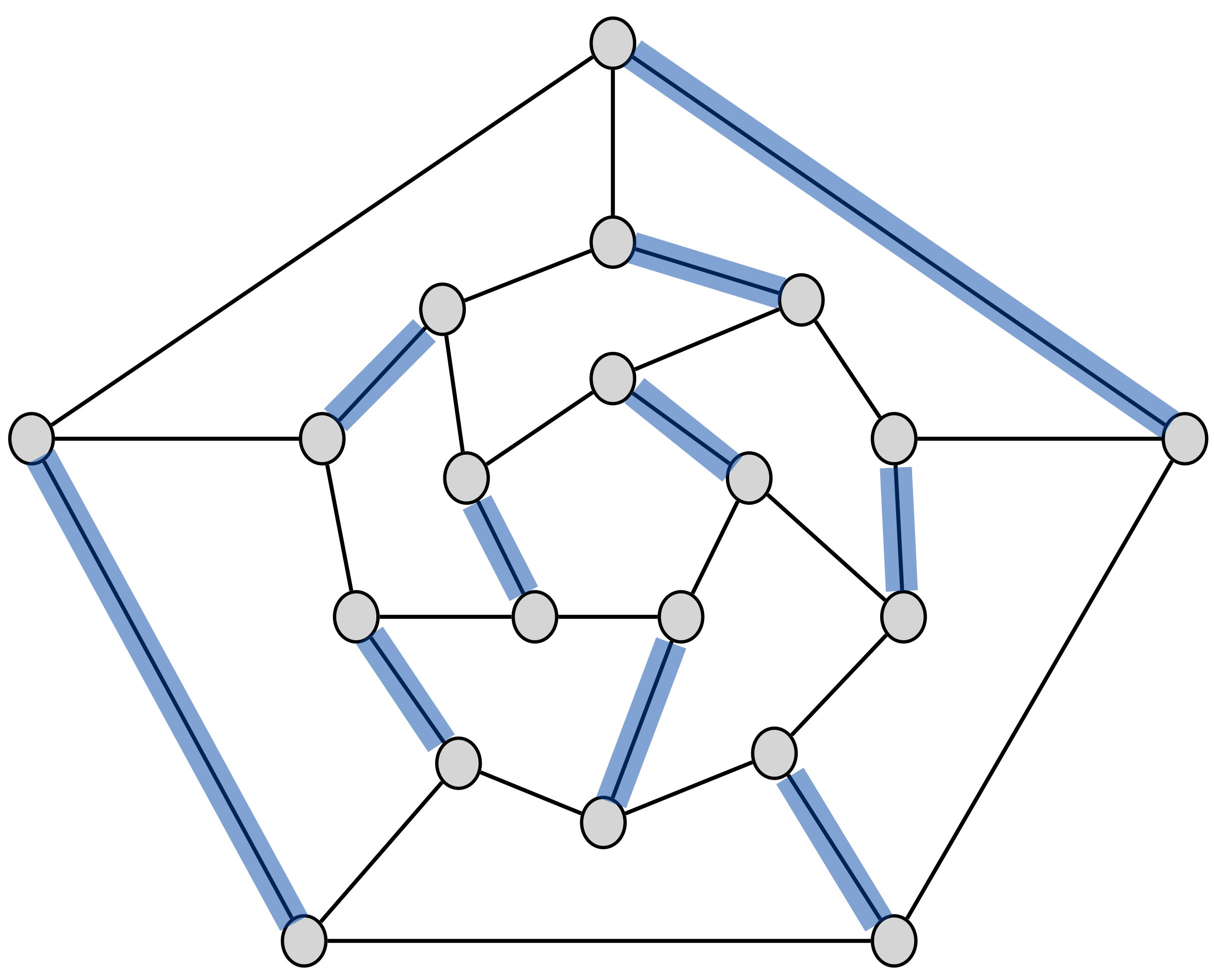
- More practice with reductions
- (Reductions will come in handy on our next topic too!)

Bipartite Matching

Review: Matching in Graphs

Definition. Given an undirected graph $G = (V, E)$, a matching $M \subseteq E$ of G is a subset of edges such that no two edges in M are incident on the same vertex.

- Said differently, a node appears as an endpoint in at most one edge in M



Review: Matching in Graphs

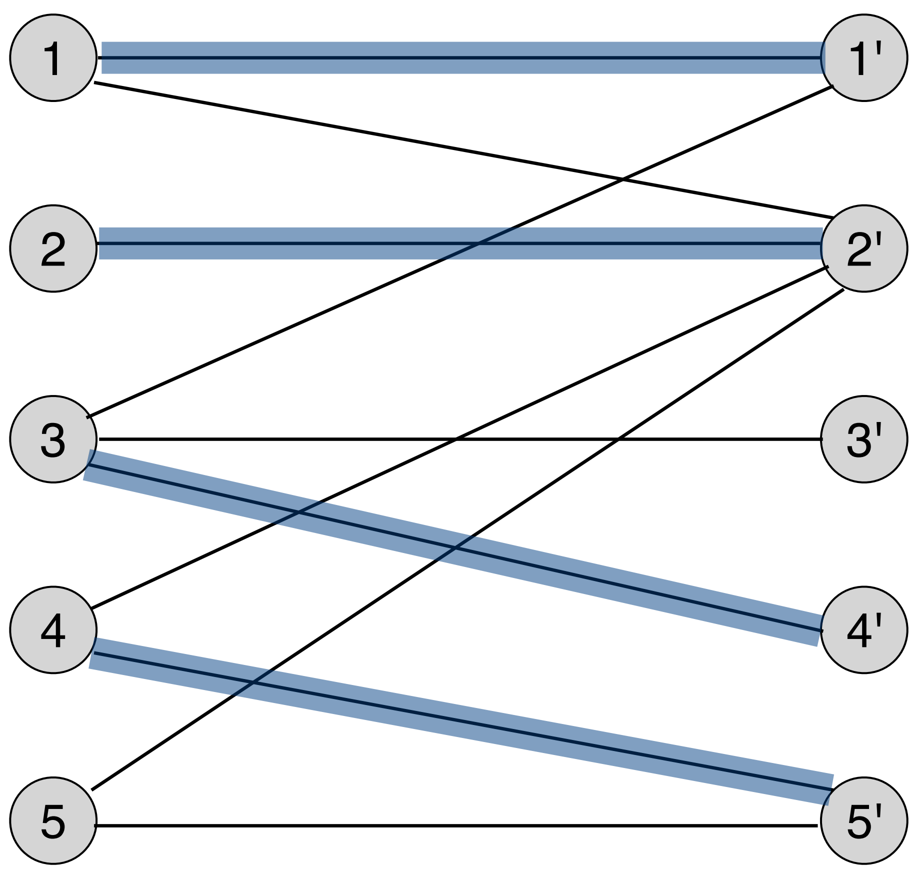
A **perfect matching** matches all nodes in G

- **Max matching problem.** Find a matching of maximum cardinality for a given graph
  - That is, a matching with maximum number of edges
  - **Observation**: If it exists, a perfect matching is maximum!

Review: Bipartite Graphs

A graph is **bipartite** if its vertices can be partitioned into two subsets X, Y such that every edge $e = (u, v)$ connects $u \in X$ and $v \in Y$

- **Bipartite matching problem.** Given a bipartite graph $G = (X \cup Y, E)$ find a maximum matching.



Bipartite Matching Examples

Can be used to model many assignment problems, e.g.:

- A is a set of jobs, B as a set of machines
- Edge (a_i, b_j) indicates where machine b_j is able to process job a_i
- Perfect matching: a way to assign each job to a machine that can process it, such that each machine is assigned exactly one job

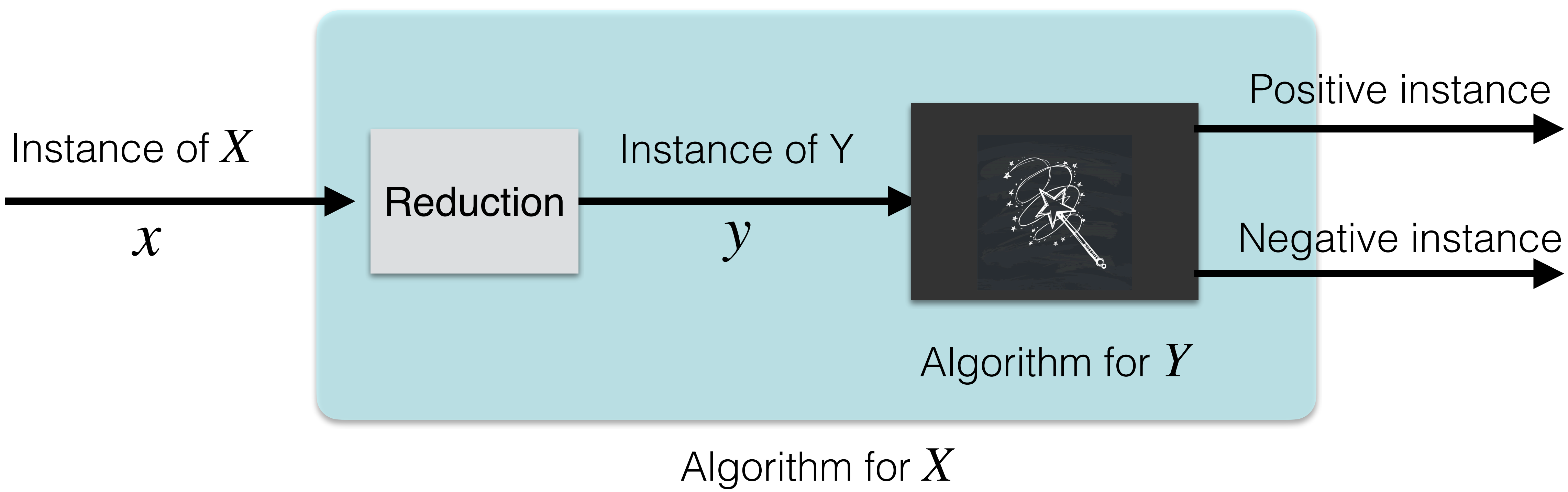
- Assigning customers to stores, students to dorms, etc.
- **Note.** This is a different problem than the one we studied for Gale-Shapely matching!

Maximum & Perfect Matchings

- One of the oldest problems in combinatorial algorithms:
    - Determine the largest matching in a bipartite graph
- This doesn't seem like a network flow problem, but we will turn it into one!
- Special case: Find a perfect matching in G if it exists
    - What conditions do we need for perfect matching?
    - Certainly need $|A| = |B|$
    - What are the necessary and sufficient conditions?
    - Will use network flow to determine

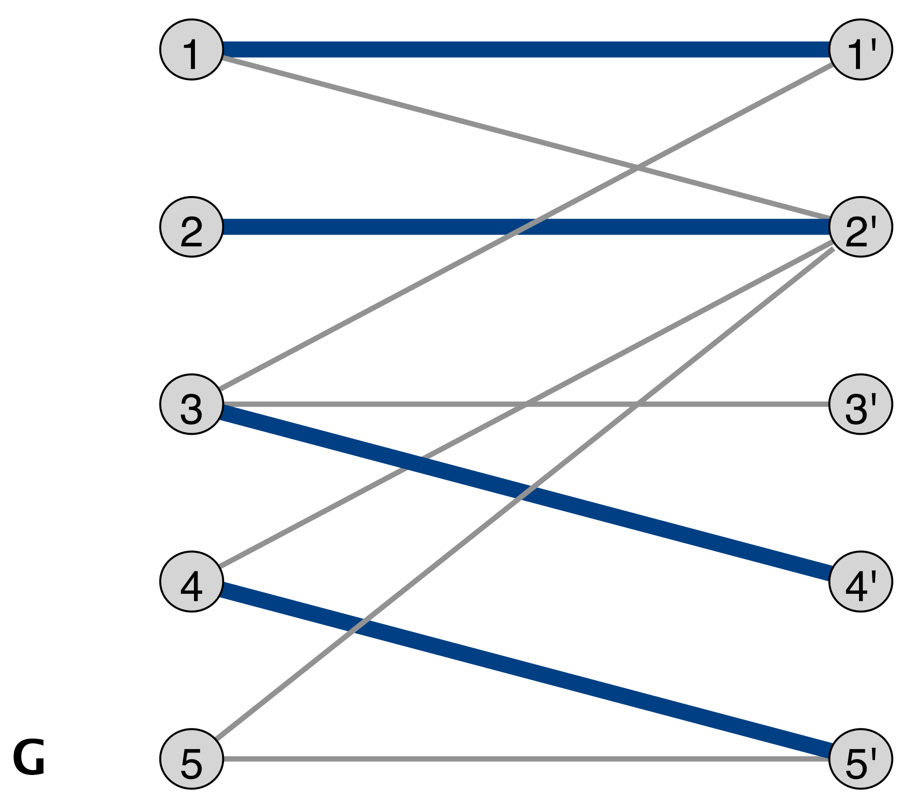
Reduction to Max Flow

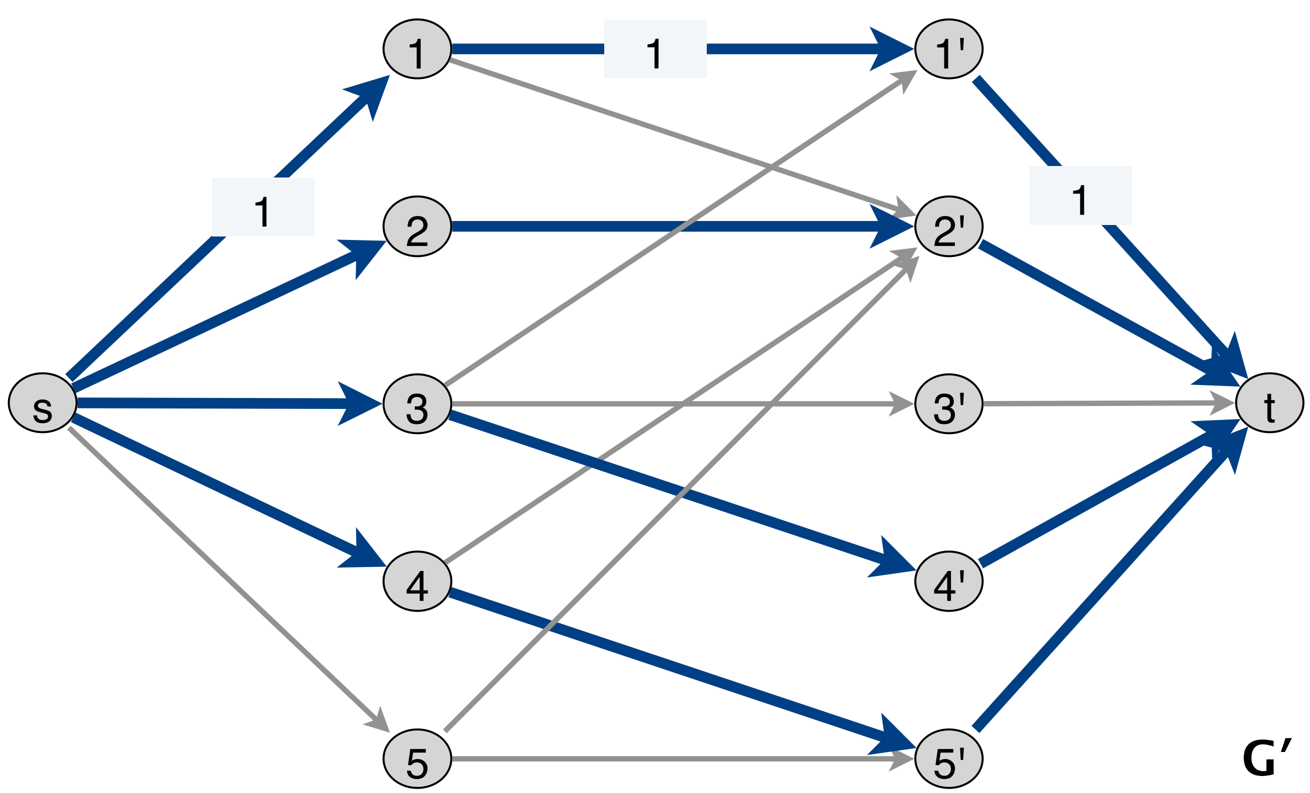
- Given arbitrary instance x of bipartite matching problem (X): A, B and edges E between A and B
- **Goal.** Create a special instance y of a max-flow problem (Y): flow network: $G(V, E, c)$, source s, sink $t \in V$ s.t.
- **1-1 correspondence.** There exists a matching of size k iff there is a flow of value k



Reduction to Max Flow

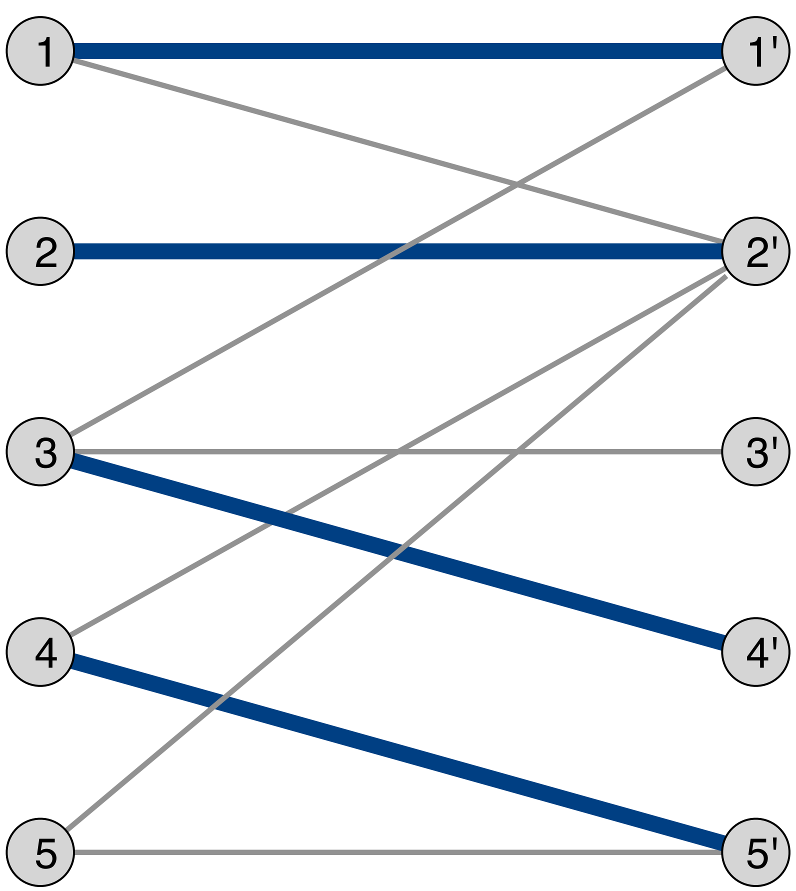
- Create a new directed graph $G' = (A \cup B \cup \{s, t\}, E', c)$
- Add edge $s \rightarrow a$ to E' for all nodes $a \in A$
- Add edge $b \rightarrow t$ to E' for all nodes $b \in B$
- Direct edge $a \rightarrow b$ in E' if $(a, b) \in E$
- Set capacity of all edges in E' to 1



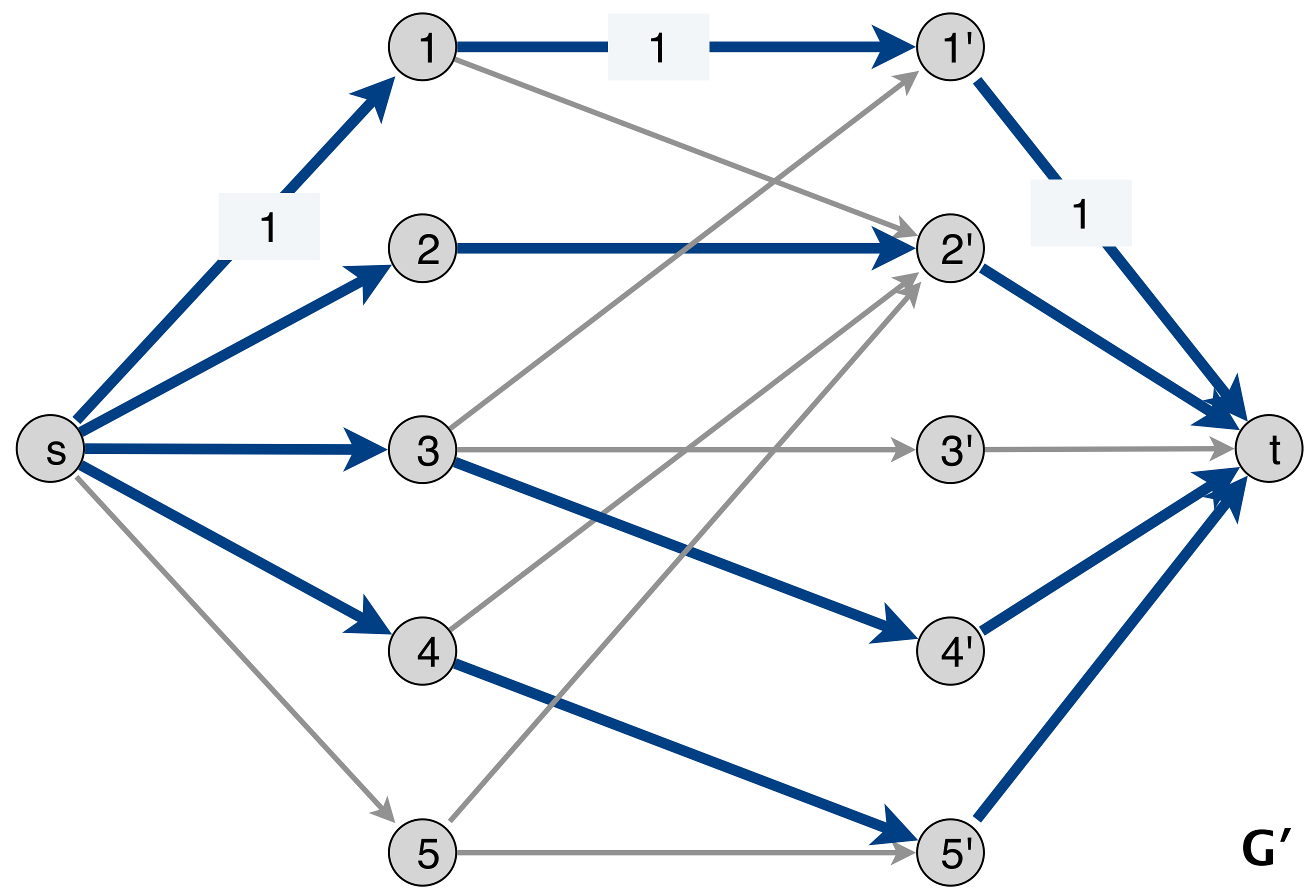


Correctness of Reduction

- **Claim** ($\Rightarrow$).
  If the bipartite graph (A, B, E) has matching M of size k then flow-network G' has an integral flow of value k.



G



G′

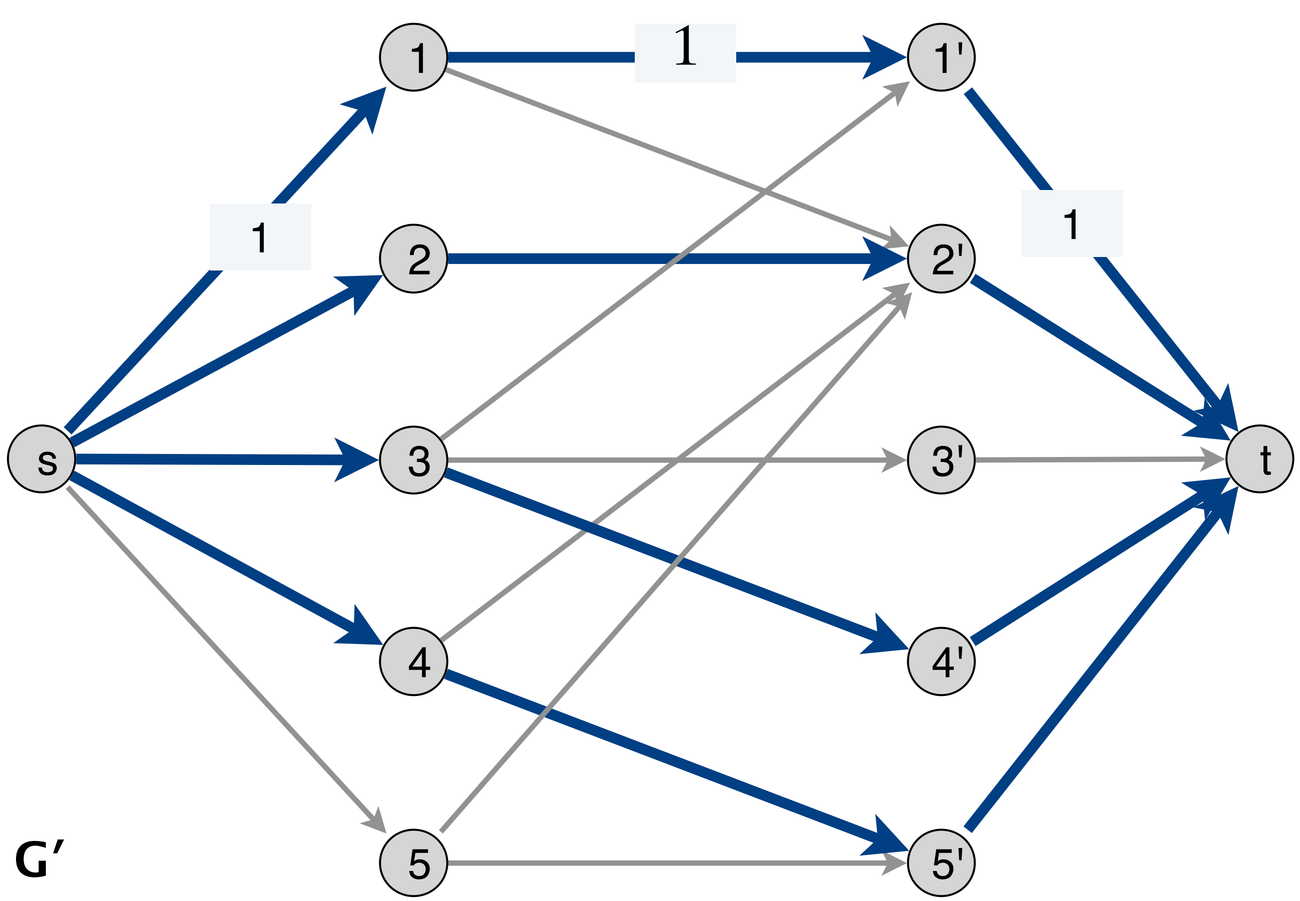
Correctness of Reduction

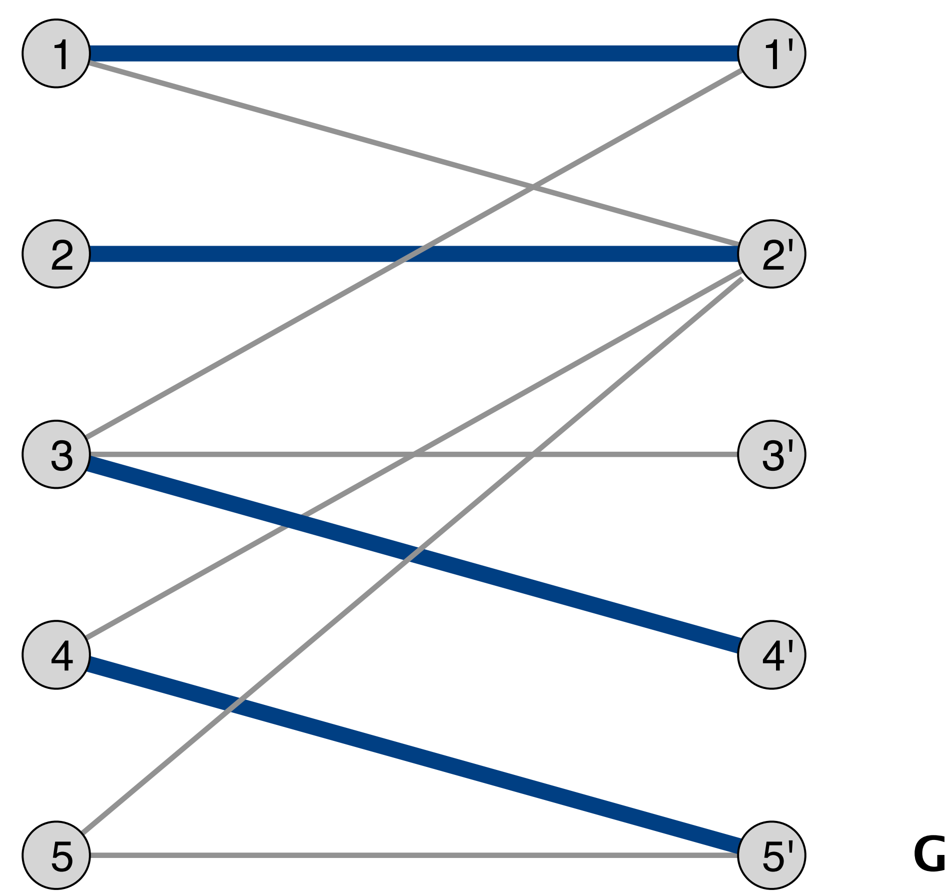
- **Claim** $(\Rightarrow)$**.**
  If the bipartite graph (A, B, E) has matching M of size k then flow-network G' has an integral flow of value k.

- **Proof** (Longer proof in textbook).
  - For every edge $e = (a, b) \in M$, let f be the flow resulting from sending 1 unit of flow along the path
    $s \to a \to b \to t$
  - f is a feasible flow (satisfies capacity and conservation) and integral
  - $v(f) = k$

Correctness of Reduction

- **Claim ($\Leftarrow$).**
  If flow-network G' has an integral flow of value k, then the bipartite graph (A, B, E) has matching M of size k.





Correctness of Reduction

- **Claim** $(\Leftarrow)$**.**
  If flow-network G' has an integral flow of value k, then the bipartite graph (A, B, E) has matching M of size k.
- **Proof**.
  - Let $M =$ set of edges from A to B with $f(e) = 1$.
  - No two edges in M share a vertex, why?
  - $|M| = k$
    - $v(f) = f_{out}(S) - f_{in}(S)$ for any $(S, V - S)$ cut
    - Let $S = A \cup \{s\}$

Edge capacities are 1

Summary & Running Time

- Proved matching of size k iff flow of value k
- Thus, max-flow iff max matching
- Running time of algorithm overall:
  - Running time of reduction + running time of solving the flow problem (dominates)
- What is running time of Ford–Fulkerson algorithm for a flow network with all unit capacities?
  - $O(nm)$
- Overall running time of finding max-cardinality bipartite matching: $O(nm)$

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)
  - Shikha Singh