“Those who cannot remember the past are condemned to repeat it.”

— Jorge Agustín Nicolás Ruiz de Santayana y Borrás

Dynamic programming
Welcome back!!!

Midterm graded but not (yet) returned

Grades were generally good!

And course grades will be curved

But… I didn’t want to release the grades until discussing what grades would correspond to particular “letters” if the semester grades were the same as the midterm grades

I will send that all in a GLOW post and then release the exams with feedback/scores on Gradescope
**Slow Recursion: Fibonacci**

**Definition.** Fibonacci numbers are defined by the following recurrence:

\[
F_n = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
F_{n-1} + F_{n-2} & \text{otherwise}
\end{cases}
\]

Recall three different implementations of Fibonacci from our previous activity:

- Naively recursive
- Local array to "memoize" the first \( n \) numbers
- Global array, worked backwards from \( n \)
The naive recursive implementation was horribly slow.

**Practice:** can we lower bound the cost?

**Step 1:** Write the recurrence

\[
T(n) = T(n - 1) + T(n - 2) + O(1)
\]
Can we lower bound the running time using techniques we already know?

\[ T(n) = T(n - 1) + T(n - 2) + \Theta(1) \]

- If we want to show that \( a \geq c \), we can show \( a \geq b \) and \( b \geq c \)

\[ T(n) \geq 2T(n - 2) + \Omega(1) \]

Let's visualize this tree!

- There are \( n/2 \) levels, each level has \( 2^i \) nodes
- Level \( i \) has cost \( \Omega(2^i) \)

\[ T(n) = \Omega(2^{n/2}) \]
Memo(r)ization

• Recursive Fibonacci algorithm is slow because it recomputes the same functions over and over

• We saw that we can speed it up considerably by writing down the results of our recursive calls, and looking them up when we need them later
Dynamic Programming: Smart Recursion

- Dynamic programming is all about smart recursion by using memoization.
- Here (fib3 from activity) we cut down on all useless recursive calls.

Dynamic Programming:  
Recursion + Memoization

**Memoization**: technique of storing expensive function call results so that they can be looked up later

- To be useful, we must carefully structure our algorithm to traverse problem space in *appropriate* order

- Memoization is a core concept of dynamic programming, but also used elsewhere
Recipe for a Dynamic Program

• **Formulate the right subproblem.** The subproblem must have an optimal substructure

• **Formulate the recurrence.** Identify how the results of the smaller subproblems can contribute to results of larger subproblems

• **State the base case(s).** The subproblem(s) so small we know the answer immediately!

• **State the final answer.** (In terms of the subproblem(s))

• **Choose a memoization data structure.** Where are you going to store already computed results? (This is often a “table”)

• **Identify evaluation order.** Identify the dependencies: which subproblems depend on which? Using these dependencies, identify an evaluation order

• **Analyze space and running time.** As always!
Weighted Scheduling

Further Reading: Chapter 6, KT
**Weighted Scheduling**

**Job scheduling.** Suppose you have a machine that can run one job at a time; \( n \) job requests, where each job \( i \) has a start time \( s_i \), finish time \( f_i \) and weight \( v_i \geq 0 \).

Each job has a weight

Overlapping jobs e.g., d and g are incompatible
Weighted Scheduling

**Input.** Given $n$ intervals labeled $1, \ldots, n$ with starting and finishing times $\{(s_1, f_1), \ldots, (s_n, f_n)\}$ and non-negative weights $\{v_1, \ldots, v_n\}$.

**Goal.** We must select compatible (non-overlapping) intervals with the maximum weight.

- That is, our goal is to find a set of intervals $I \subseteq \{1, \ldots, n\}$ that are pairwise non-overlapping and that maximize $\sum_{i \in I} v_i$. 
Remember Greedy?

- In Unweighted, *earliest-finish-time first* was optimal greedy algorithm
  - Consider jobs in order of finish times
  - Greedily pick jobs that are non-overlapping
- We proved greedy is optimal when all weights are 1
- How about the weighted interval scheduling problem?

Greedy fails spectacularly!

![Diagram showing the failure of the greedy algorithm](image-url)
Different Greedy?

We saw that not it is important to choose the right thing to “be greedy” over. Should we just pick other optimization criteria?

- **New idea**: greedily select intervals with the maximum weights, remove overlapping intervals

- Does this work?

![Greedy fails spectacularly!](image-url)
Let’s Think Recursively

The heart of dynamic programming is recursively thinking.

• Coming up with a **smaller subproblem** that has the **same optimal structure** as the original problem

• First, let’s focus on the total **value** of the optimal solution, rather than the actual set of intervals. That is,

• **Optimal value:**
  The largest $\sum_{i \in I} v_i$ where intervals in $I$ are compatible.

• Let’s also define $\text{Opt-Schedule}(n)$ to be the **value** of the optimal schedule that considers the first $n$ intervals
Let’s Think Recursively

Consider the last interval: *it’s either in the optimal solution or it’s not.*

- Whatever the optimal solution is, we can find it by considering both cases (in or out) and taking their maximum weight.

- **Case 1.** Last interval **is not** in the optimal solution
  - Remove the last interval.
    *We now have a smaller subproblem!*

- **Case 2.** Last interval **is** in the optimal solution
  - Anything that overlaps with this interval cannot be in the solution. Remove them.
    *We now have a smaller subproblem!*
Formalize the Subproblem

\textbf{Opt-Schedule}(i): value of the optimal schedule that only considers intervals \{1,\ldots, i\}, for \(0 \leq i \leq n\)
Base Case & Final Answer

**Opt-Schedule**$(i)$: value of the optimal schedule that only considers intervals \{1,..., $i$\}, for $0 \leq i \leq n$

**Base Case.**  Opt-Schedule$(0) = 0$

**Goal** (Final answer.)  Opt-Schedule$(n)$
Recurrence

How do we go from one subproblem to the next?

- The recurrence describes how to compute $\text{Opt-Schedule}(i)$ by using values of $\text{Opt-Schedule}(j)$ where $j < i$

**Case 1.** Say interval $i$ is not in the optimal solution, can we write the recurrence for this case?

- $\text{Opt-Schedule}(i) = \text{Opt-Schedule}(i - 1)$
Recurrence

How do we go from one subproblem to the next?

- The recurrence describes how to compute $\text{Opt-Schedule}(i)$ by using values of $\text{Opt-Schedule}(j)$ where $j < i$

**Case 2.** Say interval $i$ is in the optimal solution, what is the smaller subproblem we should recurse on for this case?

- No interval $j < i$ that overlaps with $i$ can be in solution
- Need to remove all such intervals to get our smaller subproblem
- How do we do that?
Suppose the intervals are sorted by finish times.

- Let $p(j)$ be the predecessor of $j$. That is, largest index $i < j$ such that intervals $i$ and $j$ are not overlapping.

- Define $p(j) = 0$ if all intervals $i < j$ overlap with $j$. 
Let \( p(j) \) be the predecessor of \( j \). That is, largest index \( i < j \) such that intervals \( i \) and \( j \) are not overlapping.

- \( p(8) = \) ?, \( p(7) = \) ?, \( p(2) = \) ?
Let $p(j)$ be the predecessor of $j$. That is, largest index $i < j$ such that intervals $i$ and $j$ are not overlapping.

- $p(8) = 1$, $p(7) = 3$, $p(2) = 0$
Recurrence

How do we go from one subproblem to the next?

- The recurrence describes how to compute $\text{Opt-Schedule}(i)$ by using values of $\text{Opt-Schedule}(j)$ where $j < i$

**Case 2.** Say interval $i$ is in the optimal solution, what is the smaller subproblem we should recurse on for this case?

  - Suppose we know $p(i)$ (the predecessor of $i$), how can we write the recurrence for this case?
  - $\text{Opt-Schedule}(i) = \text{Opt-Schedule}(p(i)) + v_i$
DP Recurrence

Opt-Schedule$(i) = \max\{\text{Opt-Schedule}(i - 1), v_i + \text{Opt-Schedule}(p(i))\}$

Optimal schedule that excludes interval $i$

Optimal schedule that includes interval $i$
Filling Out the DP Table

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**Diagram:**

- **Time:** 0, 1, 2, 3, 4
- **Tasks:**
  - Task 7: Duration 3, Starts at 0
  - Task 10: Duration 1, Starts at 0
  - Task 8: Duration 4, Starts at 2
  - Task 1: Duration 2, Starts at 0

**Legend:**
- Gray bars represent completed tasks.
- Red numbers indicate task numbers.
- Black numbers represent task durations.
Filling Out the DP Table

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time
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0 10 10 10
0 1 2 3 4
```

```
0 10 10 10
0 1 2 3 4
```

- Time: 0 1 2 3 4
- Events:
  - 7
  - 10
  - 1
  - 2
  - 8
  - 4
Filling Out the DP Table

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Time line:

- 0 to 10: Task 1 (10 units)
- 10 to 11: Task 2 (2 units)
- 11 to 13: Task 3 (3 units)
- 13 to 17: Task 4 (4 units)
Summary of DP

• **Subproblem.** Formulate the optimal substructure
  
  • For $0 \leq i \leq n$, let $\text{Opt-Schedule}(i)$ be the value of the optimal schedule that only uses intervals $\{1, \ldots, i\}$

• **Recurrence.** How to go from one subproblem to the next
  
  • $\text{Opt-Schedule}(i) = \max \{\text{Opt-Schedule}(i - 1), v_i + \text{Opt-Schedule}(p(i))\}$

• **Base case.** The problem(s) we immediately know the answer to.
  
  • $\text{Opt-Scheduler}(0) = 0$ (no intervals to schedule)

• **Correctness.**
  
  • Use induction based on the recurrence
Remaining Pieces

- Final answer in terms of subproblem?
  - Opt-Schedule\[n\]
- Evaluation order (in what order can be fill the DP table)
  - \(i = 0 \rightarrow n\), start with base case and use that to fill the rest
- Memoization data structure: 1-D array
- Final piece:
  - Running time and space
    - Space: \(O(n)\)
  - Time: preprocessing + time to fill array
Computing $p[i]$ (Preprocessing)

- How quickly can we compute $p[i]$?
  - We could do a linear scan for each $i$: $O(i)$ per interval
  - This would be $O(n^2)$ overall...
- What if we had intervals sorted by their finish time $F[1,\ldots,n]$?
  - For each interval, we could binary search over $F[1,\ldots,n]$ to find the first $j < i$ such that $f_j \leq s_i$
  - Binary searching would take $O(\log n)$ per interval, $O(n \log n)$ total
- Time $O(n \log n)$ to compute the array $p[]$
  - This covers sorting + binary searching
Running Time

• How many subproblems do we need to solve?
  • $O(n)$

• How long does it take to solve a subproblem?
  • $O(1)$ to take the max

• Preprocessing time:
  • Need to sort; $O(n \log n)$
  • Need to find $p(i)$ for all each $i$: $O(n \log n)$
  • Overall: $O(n \log n) + O(n) = O(n \log n)$

Wait!!! We’ve only computed the value, not the actual interval set!!!
Recreating Chosen Intervals

Suppose we have $M[]$ of optimal weights.

- **Big Question**: How can we reconstruct the optimal set of intervals?

Identifying which of the two cases was larger tells us whether or not interval $i$ was included:

$$\text{Opt-Schedule}(i) = \max \{ \text{Opt-Schedule}(i-1), v_i + \text{Opt-Schedule}(p(i)) \}$$

This value is bigger: $i$ not in OPT

This value is bigger: $i$ is in OPT
Recursive Solution?

Suppose for now that we do not memoize: just a divide and conquer recursion approach to the problem.

Opt-Schedule\((j)\):

- If \( j = 0 \), return 0
- Else
  - Return \( \max(\text{Opt-Schedule}(j - 1), v_j + \text{Opt-Schedule}(p(j))) \)

- How many recursive calls in the worst case?
  - Depends on \( p(i) \)
- Can we create a really bad instance?
Recursive Solution: Exponential

• For this example, asymptotically how many recursive calls?

• Grows like the Fibonacci sequence (exponential):
  \[ T(n) = T(n - 1) + T(n - 2) + O(1) \]

• Lots of redundancy!
  • How many distinct subproblems are there to solve?
  • Opt-Schedule(i) for \( 1 \leq i \leq n + 1 \)

\[ p(1) = 0, \ p(j) = j - 2 \]

recursion tree
Dynamic Programming Tips

• Recurrence/subproblem is the key!

• DP is a lot like divide and conquer, while writing extra things down

• When coming to a new problem, ask yourself what subproblems may be useful? How can you break that subproblem into smaller subproblems?

• Be clear while writing the subproblem and recurrence!

• In DP we usually keep track of the cost of a solution, rather than the solution itself
Recipe for a Dynamic Program

• **Formulate the right subproblem.**  The subproblem must have an optimal substructure

• **Formulate the recurrence.**  Identify how the result of the smaller subproblems can lead to that of a larger subproblem

• **State the base case(s).**  The subproblem thats so small we know the answer to it!

• **State the final answer.**  (In terms of the subproblem)

• **Choose a memoization data structure.**  Where are you going to store already computed results? (Usually a table)

• **Identify evaluation order.**  Identify the dependencies: which subproblems depend on which ones? Using these dependencies, identify an evaluation order

• **Analyze space and running time.**  As always!
Acknowledgments

• Some of the material in these slides are taken from
  • Jeff Erickson’s Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)