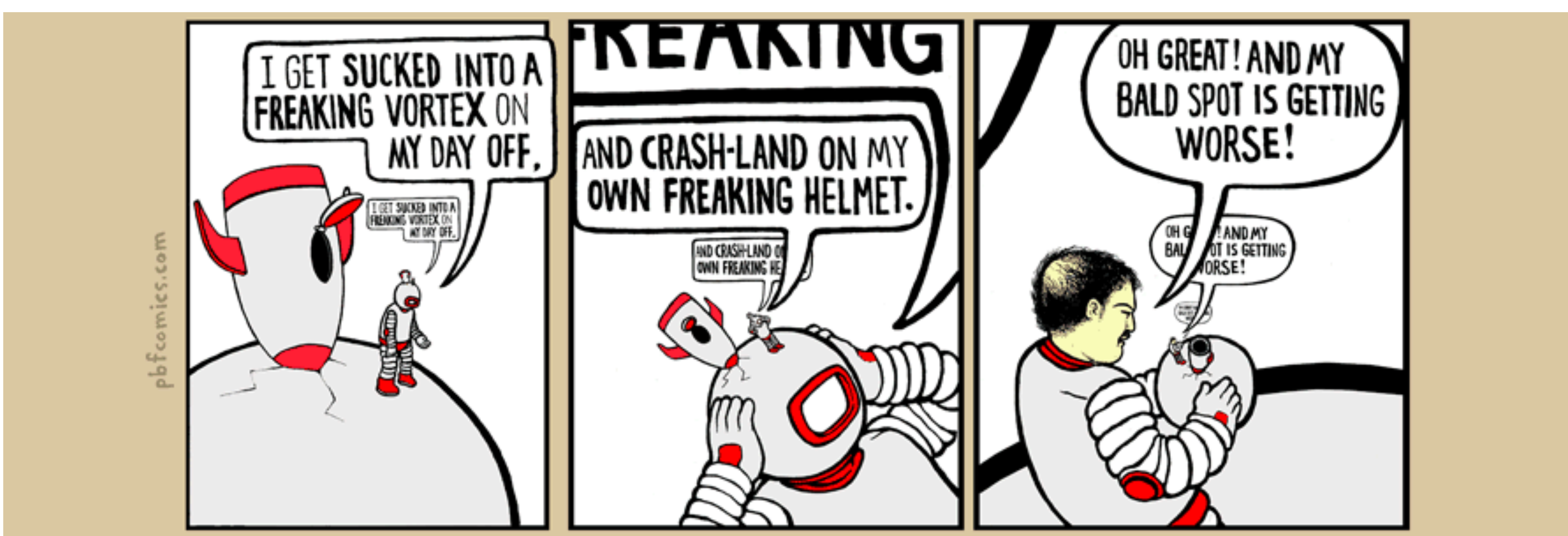
Divide and Conquer: Sorting and Recurrences

Divide & Conquer: The Pattern

- **Divide** the problem into several independent smaller instances of exactly the same problem
- **Delegate** each smaller instance to the **Recursive Leap of Faith** (technically known as induction hypothesis)
- **Combine** the solutions for the smaller instances



Review: Merge Sort

MergeSort(L):

if L has one element
    return L

Base case

Divide L into two halves A and B

$A \leftarrow$ **MergeSort**(A)

$B \leftarrow$ **MergeSort**(B)

Recursive leaps of faith

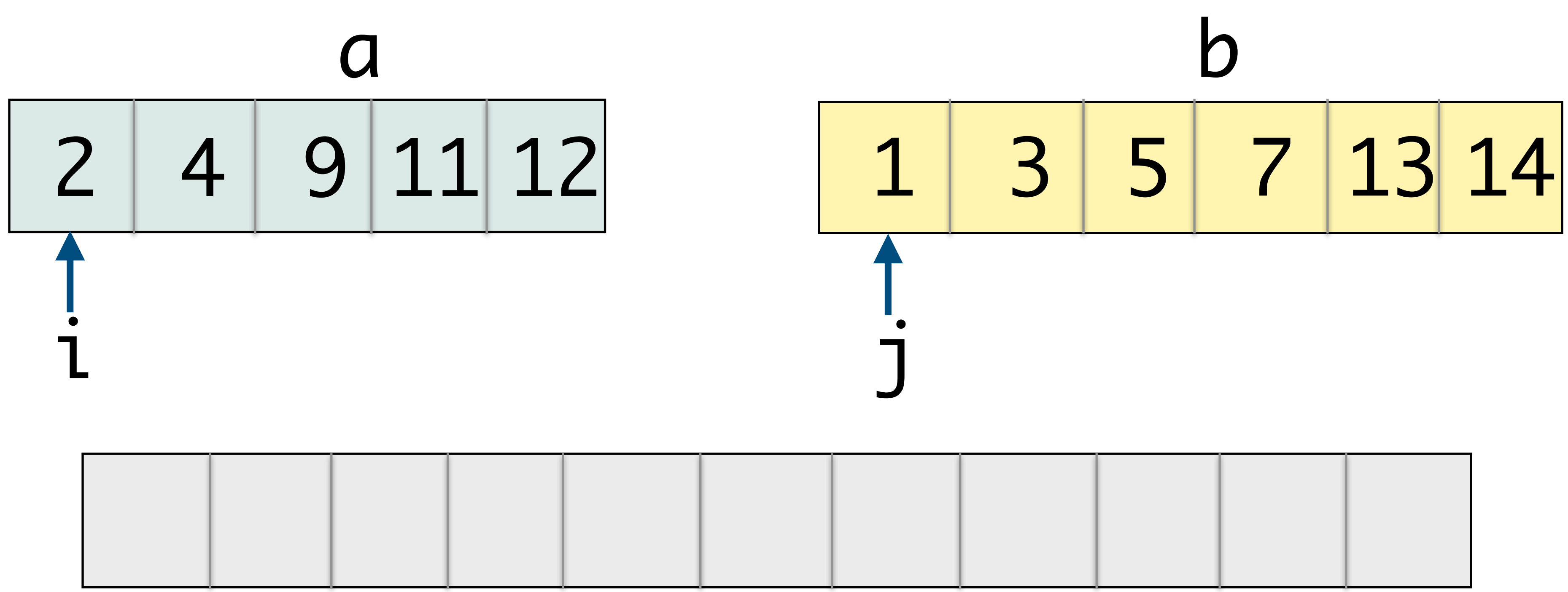
$L \leftarrow$ **Merge**(A, B)

Combine solutions

return L

Merge Step: $\Theta(n)$

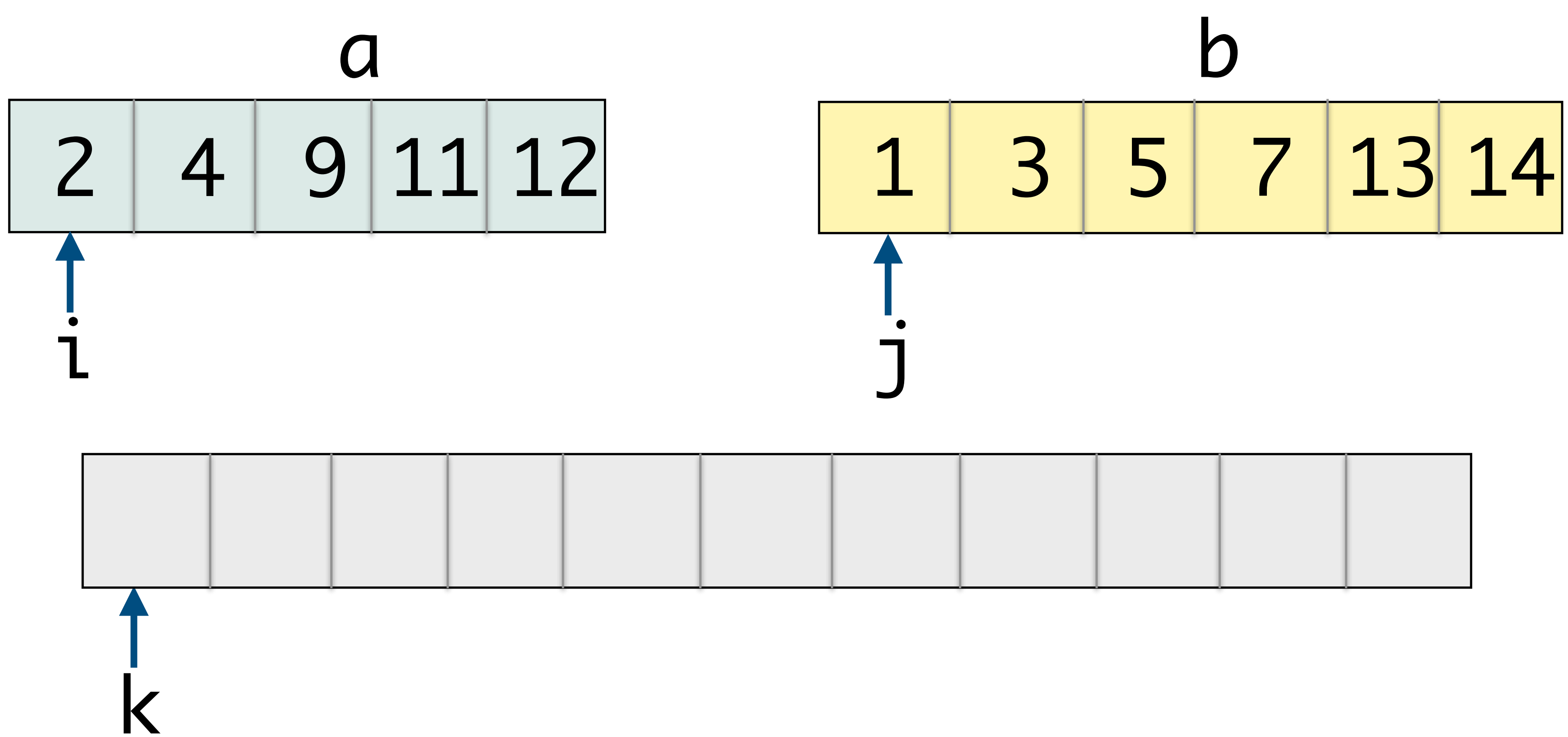
- Scan sorted lists from left to right
- Compare element by element; create new merged list



Merge Step: $\Theta(n)$

`Is a[i] <= b[j] ?`

- `Yes, a[i] appended to c, advance i`
- `No, b[j] appended to c, advance j`



merged list c

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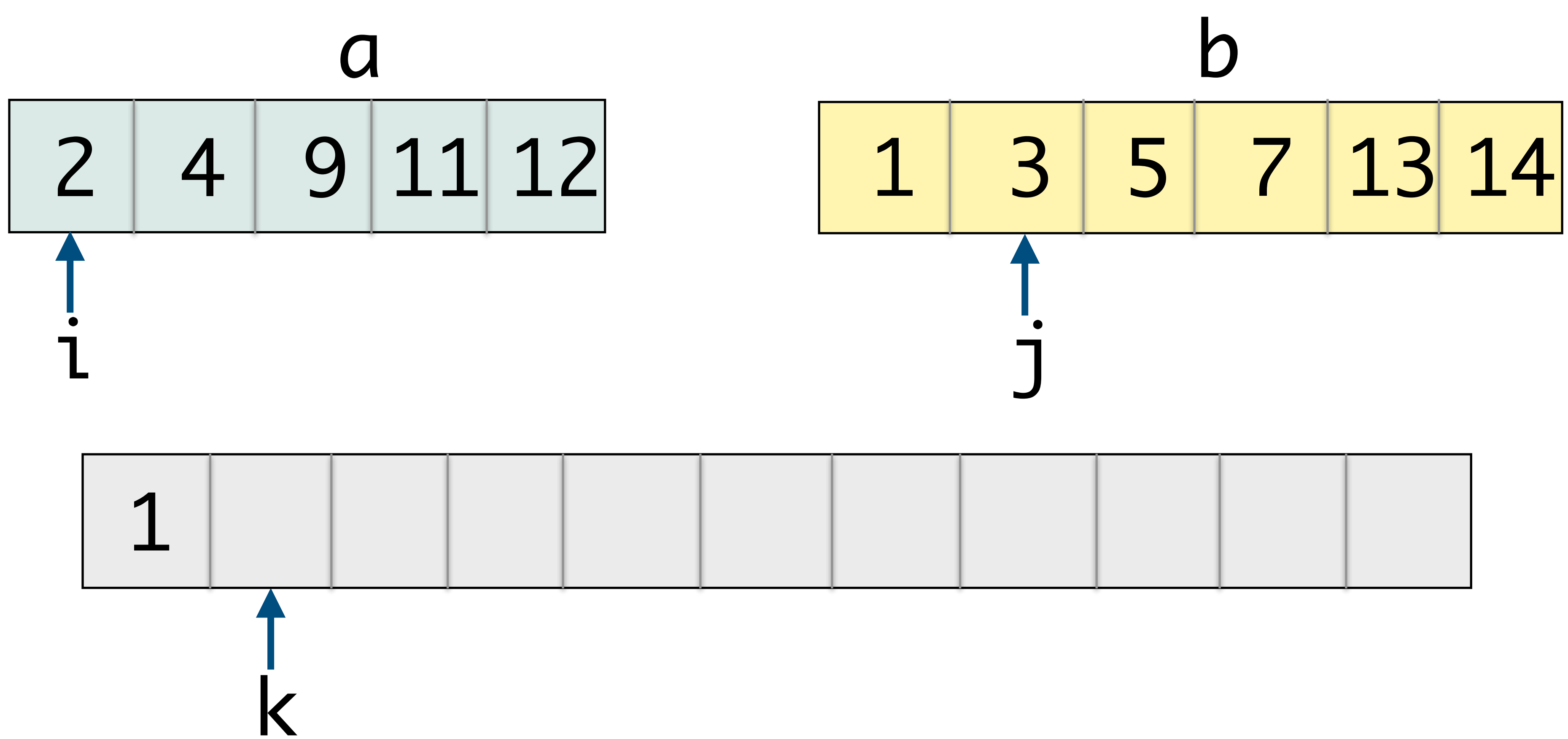

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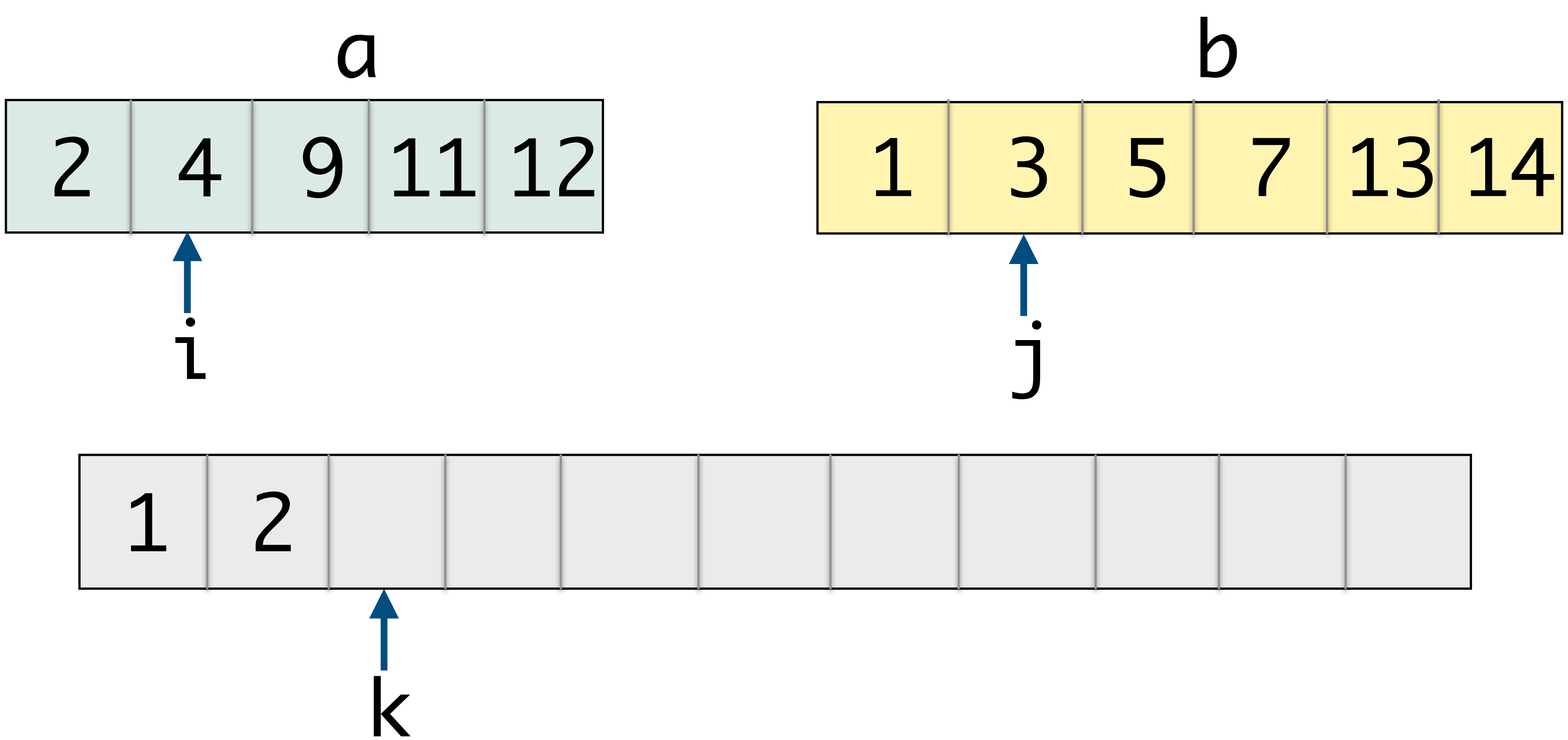

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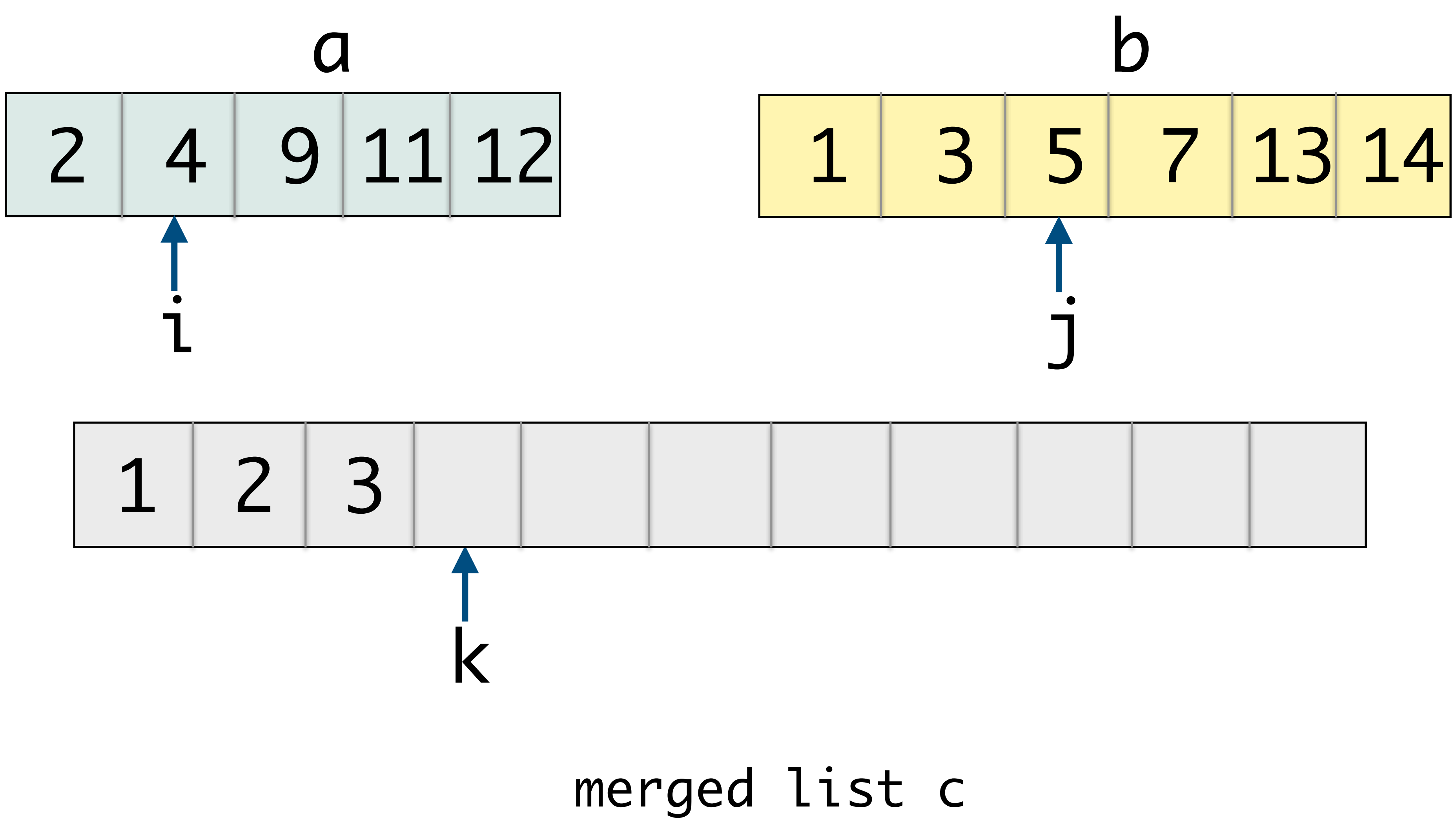

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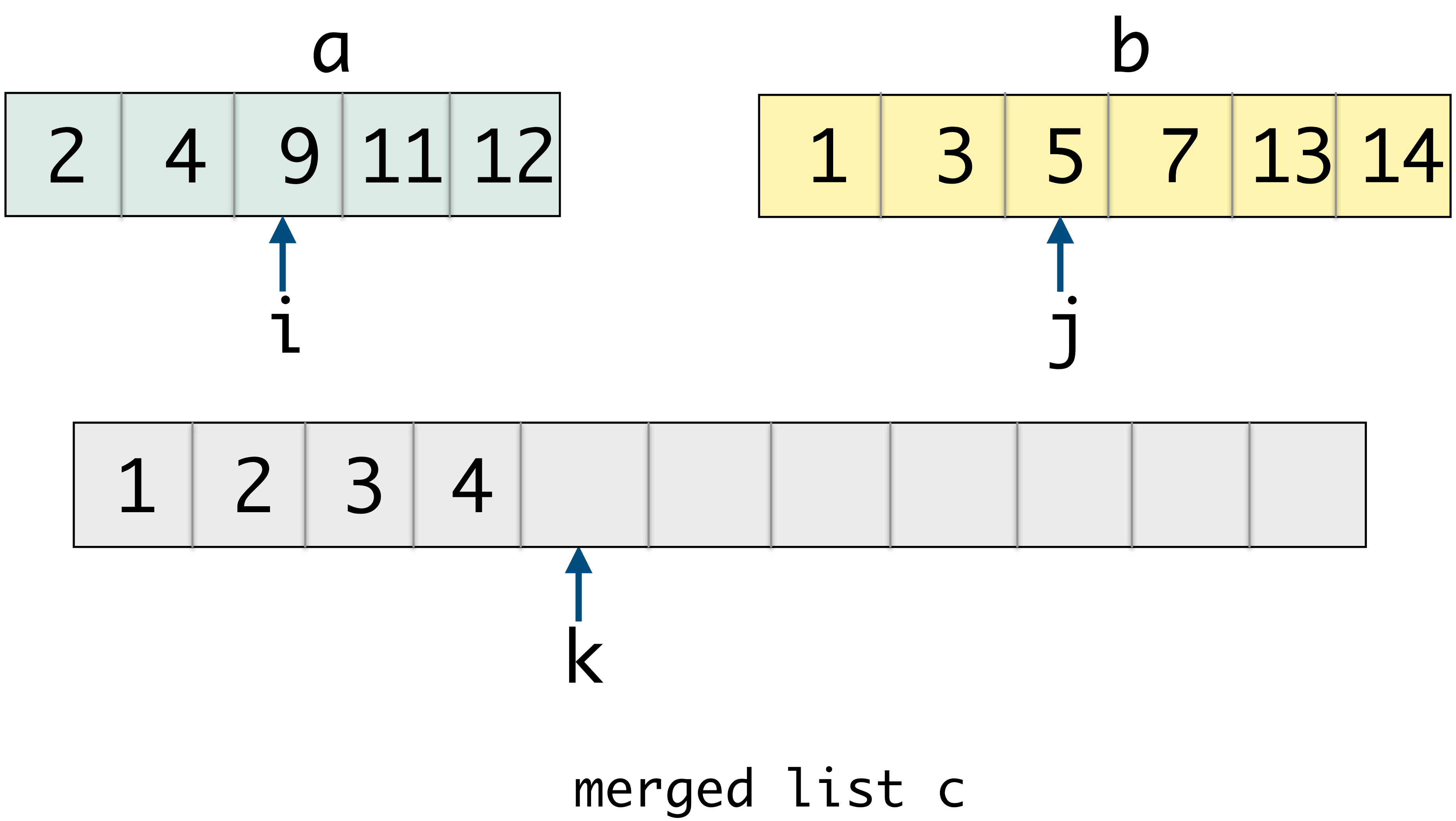

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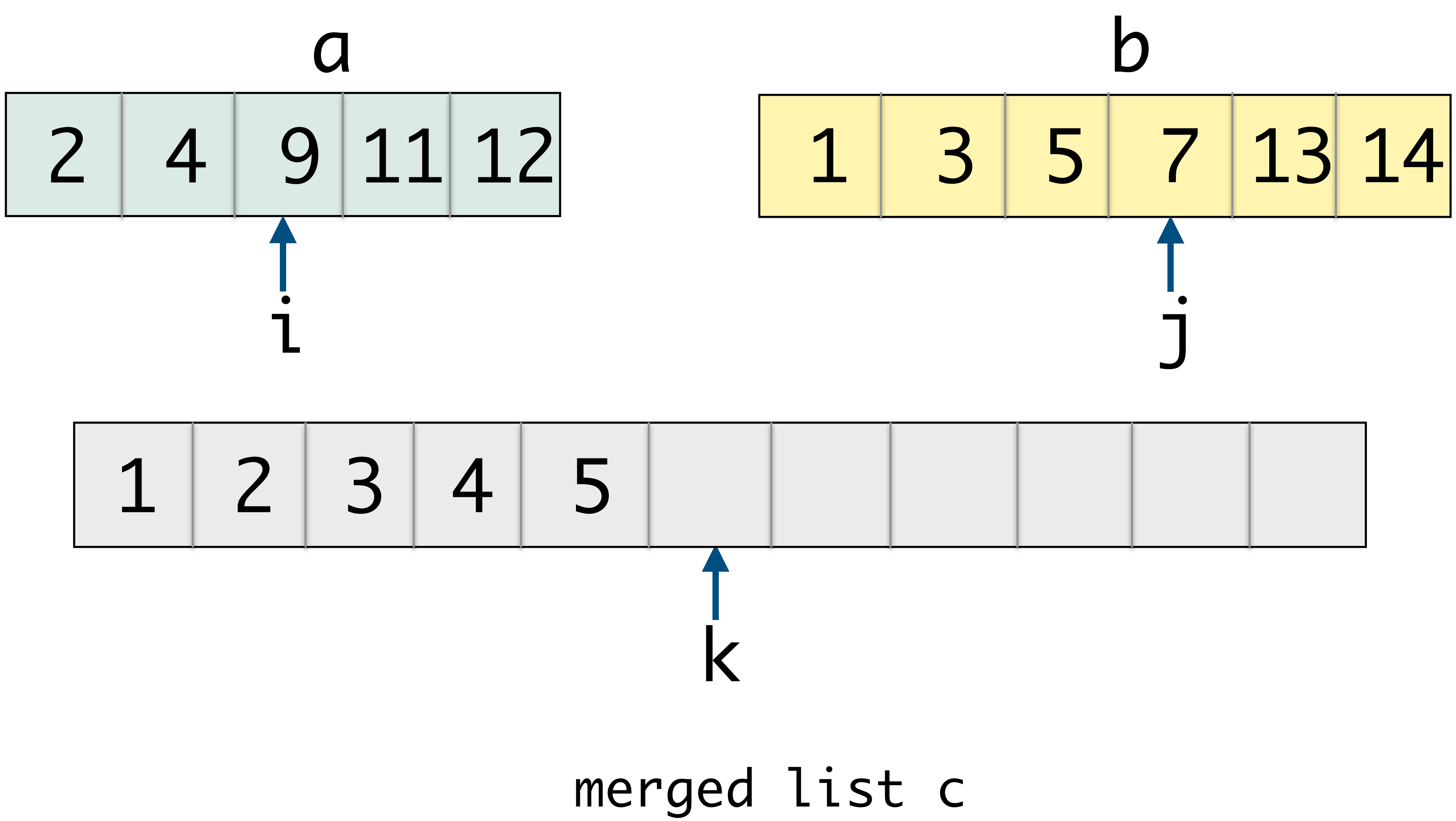

Yada yada yada…

Merge Step: $\Theta(n)$

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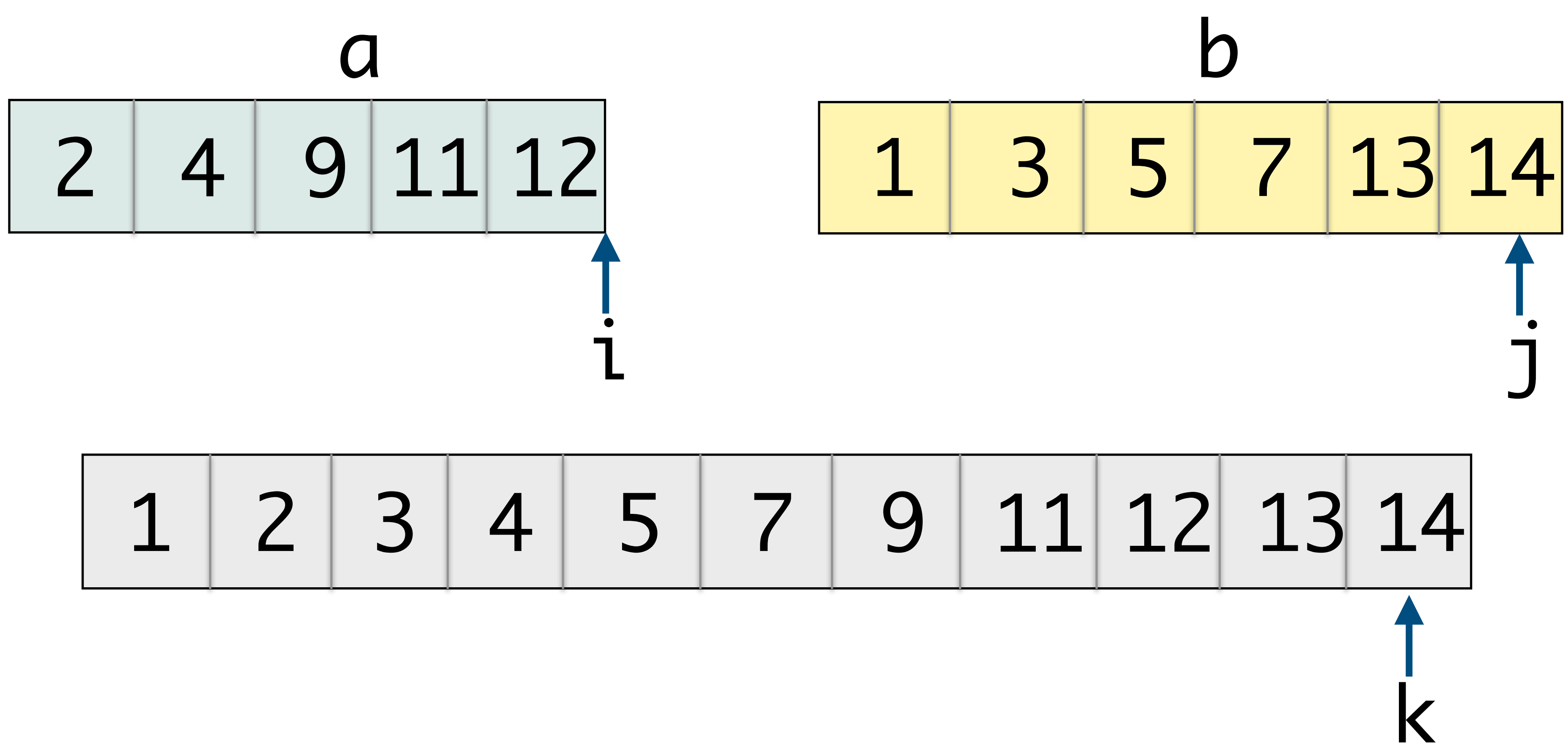


merged list c

Correctness: D&C Algorithms

- **Proving Correctness** (often follow proof by induction pattern)
  - Show base case holds
  - Assume your recursive calls return the correct solution (induction hypothesis)
  - Inductive step: crux of the proof
    - Must show that the solutions returned by the recursive calls are **"combined"** correctly

Correctness Sketch: Merge Sort

- **Claim. (Combine step.)** Merge subroutine correctly merges two sorted subarrays $A[1,\ldots i]$ and $B[1,\ldots,j]$ where $i+j=n$.
  - Will prove that for the first k iterations of the loop, correctly merges A and B (from $n=0$ to $n=k$).
- **Invariant:** Merged array is sorted after every iteration.
- Base case: $k=0$
  - Algorithm correctly merges two empty subarrays
- For inductive step, there are multiple cases, including $a_i \leq b_j$, $a_i > b_j$
  - for each case, must show that newly added element maintains sorted-ness

Analyzing Running Time

- For this topic, our main focus will be on analysis of running time
- We analyze the running time of recursive functions by:
  - **Considering the recursive calls**: both the number of calls made and the size of the inputs to each call
    - e.g., merge sort on an input of size n makes two recursive calls each on an input of size $n/2$
  - **The time spent "combining"** solutions ("non-recursive work") returned by recursive calls
    - e.g. merge step combines the sorted arrays in $\Theta(n)$ time
- Using the two, we typically write a **running time recurrence**

Running Time Recurrence

- Let $T(n)$ represent the worst-case running time of merge sort on an input of size n
- $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n)$
- **Base case:** $T(1) = 1$; often ignored
- We will ignore the floors and ceilings (turns out it doesn't matter for asymptotic bounds; we'll show this later)
- So the recurrence simplifies to:
    - $T(n) = 2T(n/2) + O(n)$
    - The answer to this ends up being $T(n) = O(n \log n)$
    - The next slides will discuss different ways to derive this

Recurrences: Unfolding

Method 1. Unfolding the recurrence

- Assume $n = 2^\ell$ (that is, $\ell = \log n$)
- Because we don't care about constant factors and are only upper-bounding, we can always choose smallest power of 2 that is greater than n. That is, $n < n' = 2^\ell < 2n$
- We can explicitly add in our constants

$T(n) = 2T(n/2) + cn$ = $2T(2^{\ell-1}) + c2^\ell$ (change of variable, replace n)

= $2(2T(2^{\ell-2}) + c2^{\ell-1}) + c2^\ell$ = $2^2T(2^{\ell-2}) + 2 \cdot c2^\ell$

= $2^3T(2^{\ell-3}) + 3 \cdot c2^\ell$

= $\ldots$

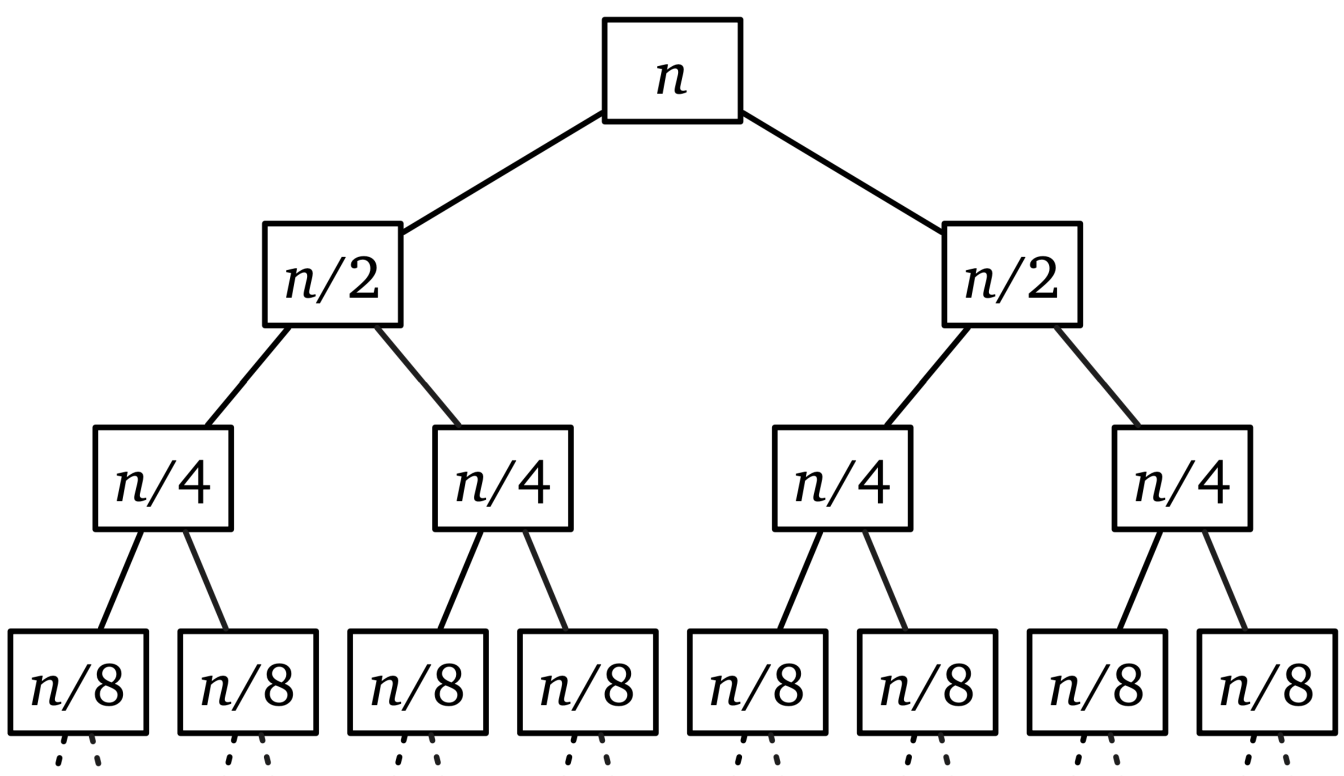
$= 2^\ell T(2^0) + c\ell 2^\ell = O(n \log n)$

Recurrences: Recursion Tree

Method 2. Recursion Trees

Recommended Method!

- Number of levels: $\log_2 n$
- Number of nodes in level i: 2^i
- Problem size at level i: $n/2^i$
- Total work done at each level: $2^i \cdot (n/2^i) = n$



Recurrences: Recursion Tree

- This is really a method of visualization
- Very similar to unrolling, but much easier to keep track of what's going on
- It's not (quite) a proof, but generally it is sufficient for reasoning about running times in this class
    - "Solve the recurrence" can be done by drawing the recursion tree and explaining the solution

Recurrences: Guess & Verify

Method 3. Guess and Verify

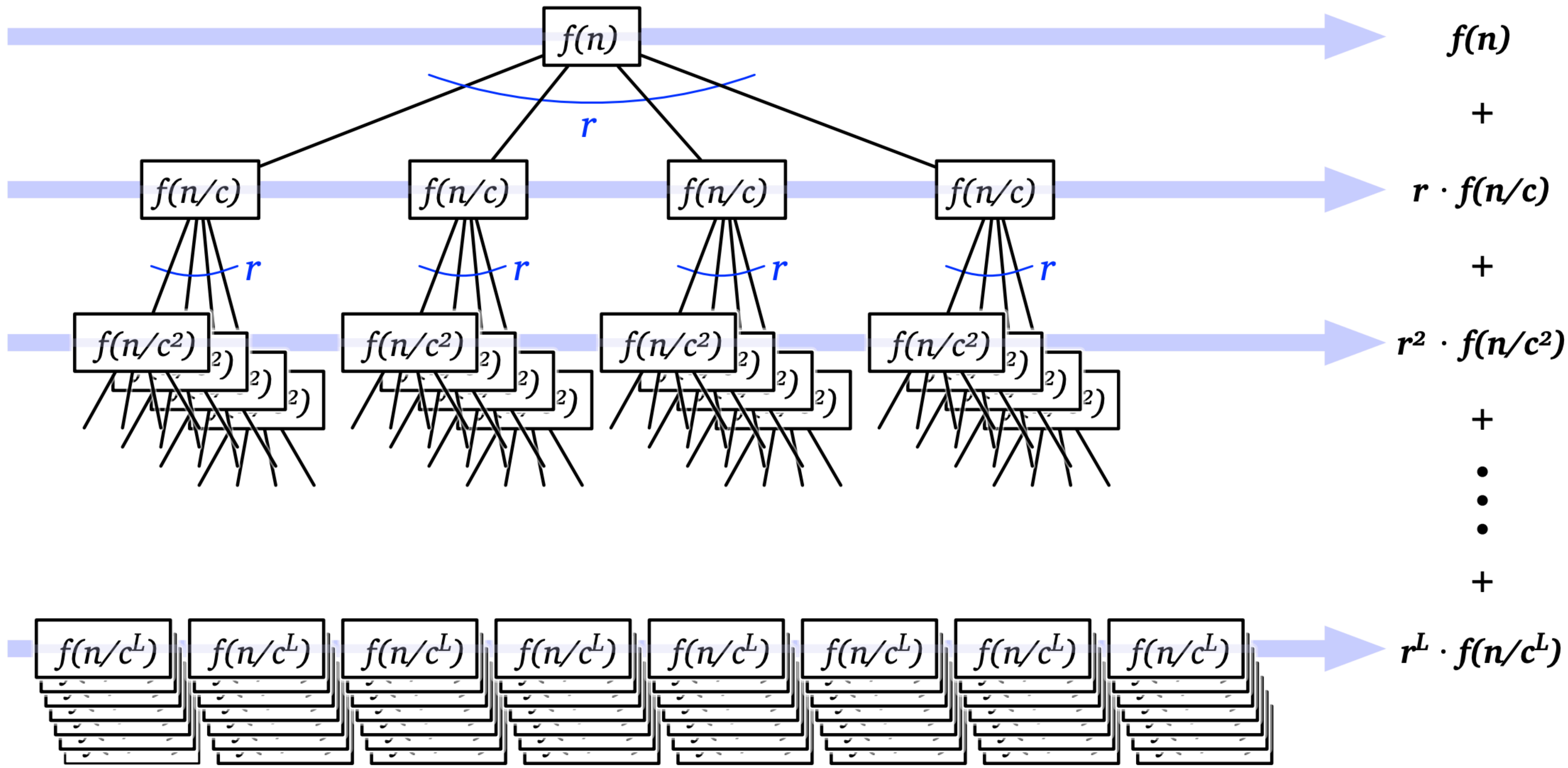
- Eyeball recurrence and make a guess
- Verify guess using induction

- More on this later…

General Recursion Trees

- Consider a divide and conquer algorithm that
  - spends $O(f(n))$ time on non-recursive work and makes r recursive calls, each on a problem of size n/c
- Up to constant factors (which we hide in $O()$), the running time of the algorithm is given by what **recurrence**?
  - $T(n) = rT(n/c) + f(n)$
- Because we care about asymptotic bounds, we can assume base case is a small constant, say $T(n) = 1$

General Recursion Trees



A recursion tree for the recurrence $T(n) = rT(n/c) + f(n)$

- For each i, the ith level of tree has exactly r^i nodes
- Each node at level i, has cost $f(n/c^i)$

General Recursion Trees

- Running time $T(n)$ of a recursive algorithm is the sum of all the values (sum of work at all nodes at each level) in the recursion tree
- The ith level of the tree has exactly r^i nodes
- And each node at level i, has cost $f(n/c^i)$

Thus, the total recurrence costs: $T(n) = \sum_{i=0}^{L} r^i \cdot f(n/c^i)$

- Here $L = \log_c n$ is the depth of the tree
- Number of leaves in the tree: $r^L = n^{\log_c r}$
- Cost at leaves: $O(n^{\log_c r} f(1))$

$$r^L = r^{\log_c n} = (2^{\log_2 r})^{\log_c n} = (2^{\log_c n})^{\log_2 r} = (2^{\log_2 n})^{\frac{\log_2 r}{\log_2 c}} = n^{\log_c r}$$

Common Cases

Don't forget: $\sum_{i=0}^{L} a^i = \frac{a^{L+1} - 1}{a - 1}$

$$T(n) = \sum_{i=0}^{L} r^i \cdot f(n/c^i)$$

- **Decreasing series.** If the series decays exponentially (every term is a constant factor smaller than previous), cost at root dominates:

$$T(n) = O(f(n))$$

- **Equal.** If all terms in the series are equal:

$$T(n) = O(f(n) \cdot L) = O(f(n) \log n)$$

- **Increasing series.** If the series grows exponentially (every term is constant factor larger), then the cost at leaves dominates:

$$T(n) = O(n^{\log_c r})$$

Master Theorem (optional)

Set of rules to solve some common recurrences automatically

(Master Theorem) Let $a \geq 1$, $b > 1$ and $f(n) \geq 0$. Let $T(n)$ be defined by the recurrence $T(n) = aT(n/b) + f(n)$ and $T(1) = O(1)$.
Then $T(n)$ can be bounded asymptotically as follows.

- If $f(n) = n^{\log_b a - \epsilon}$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- If $f(n) = \Omega(n^{\log_b a + \epsilon})$, for some constant $\epsilon > 0$, and if $af(n/b) \leq c_0 f(n)$ for some constant $c_0 < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

Master Theorem

- It exists; it can make things easier. You don't need to know it
- OK to use in this class, but I don't encourage (nor discourage) it
  - Recursion trees promote a better understanding of the recurrence—and they can be simpler
- Master Theorem only applies to some recurrences (generalizations do exist)

Divide and Conquer: Sorting and Recurrences

Divide & Conquer: Quicksort

Who remembers Quicksort?

- Choose a pivot element from the array
- Partition the array into two parts:
  - LEFT: all elements that are less than or equal to the pivot
  - RIGHT: all elements that are greater than the pivot
- Recursively quicksort the LEFT and RIGHT subarrays

Input:	S	O	R	T	I	N	G	E	X	A	M	P	L
Choose a pivot:	S	O	R	T	I	N	G	E	X	A	M	**P**	L
Partition:	A	G	O	E	I	N	L	M	**P**	T	X	S	R
Recurse Left:	A	E	G	I	L	M	N	O	**P**	T	X	S	R
Recurse Right:	A	E	G	I	L	M	N	O	**P**	R	S	T	X

Divide & Conquer: Quicksort

- **Description.** (Divide and conquer): often the cleanest way to present is **short and clean pseudocode** with high level explanation
- **Correctness proof.** Induction and showing that partition step correctly partitions the array.

```
QuickSort(A[1..n]):
  if (n > 1)
      Choose a pivot element A[p]
      r ← Partition(A, p)
      QuickSort(A[1..r−1])      ⟨⟨Recurse!⟩⟩
      QuickSort(A[r+1..n])      ⟨⟨Recurse!⟩⟩
```

Quick Sort Analysis

- How long does partition take? $O(n)$
- Let's write a recurrence relation for quick sort!
- Challenge: the size of the subproblems depends pivot!?!?!
  - **Idea**: let r be the rank of the pivot, where rank is the (lowest) index of the item in the sorted list.
  - Base case:

$$T(1) = 1$$

Partition that's $\leq$ pivot

Partition that's $>$ pivot

  - General Case:

$$T(n) = T(r-1) + T(n-r) + O(n)$$

Quick Sort Analysis

- Let us analyze some cases for r
  - **Best case:**
    - r is the median: $r = \lfloor n/2 \rfloor$
      - (we can show how to compute the median in $O(n)$ time)
  - **Worst case:**
    - $r = 1$ or $r = n$
      - When everything falls on "one side" of the pivot
  - **Something in between:**
    - say $n/10 \leq r \leq 9n/10$

Note in the worst-case analysis, we would only consider the worst case for r.
We will look at the different cases to get a sense and get some practice.

Quick Sort: Cases

- Suppose $r = n/2$ (pivot is the median element), then recurrence is:
  - $T(n) = 2T(n/2) + O(n)$, $T(1) = 1$
    - We have already solved this recurrence!
    - $T(n) = O(n \log n)$

- Suppose $r = 1$ or $r = n - 1$, then the recurrence is:
  - $T(n) = T(n-1) + T(1) + O(n), T(1) = 1$
  - What running time would this recurrence lead to?
    - Let's draw the recurrence tree…
    - $T(n) = \Theta(n^2)$ (notice: this is tight!)

Quick Sort: Cases

- Suppose $r = n/10$ (that is, you get a one-tenth, nine-tenths split)
  - What is the recurrence?
    - $T(n) = T(n/10) + T(9n/10) + O(n)$
  - Let's look at the recursion tree for this recurrence…
- We get $T(n) = O(n \log n)$, in fact, we get $\Theta(n \log n)$
- In general, the following holds (we'll show it later):
  - $T(n) = T(\alpha n) + T(\beta n) + O(n)$
    - If $\alpha + \beta < 1 : T(n) = O(n)$
    - If $\alpha + \beta = 1 : T(n) = O(n \log n)$

Quick Sort: Theory and Practice

- We can find the **median element in $\Theta(n)$** time
  - Using divide and conquer!
  - But in practice, the constants hidden in the Oh notation for median finding are too large to use for sorting
- Common heuristic
  - Median of three (pick elements from the start, middle and end and take their median)
- If the pivot is chosen **uniformly at random**
  - quick sort runs in time $O(n \log n)$ in expectation and *with high probability*
  - We will prove this in the second half of the class

Recurrences

So far we've focused on divide and conquer algorithms, where we split the problem in more than one subproblem.

Question. Can you think of some examples (that you haven't seen so far) where we split the problem into **one** smaller subproblem?

D&C: One Smaller Subproblem

- Binary search in array
  - $T(n) = T(n/2) + 1$
- Search in a binary search tree
  - $T(n) = T(n/2) + 1$
- Fast exponentiation (you may not have seen this)
  - Compute a^n, how many multiplications?
  - Naive way: $a \cdot a \cdot \ldots \cdot a$ (n times)
  - Faster way: $a^n = (a^{n/2})^2$ (suppose n is even)
  - $T(n) = T(n/2) + 1$
  - What does this solve to?

Selection

Selection: Problem Statement

Given an array $A[1,\ldots,n]$ of size n, find the kth smallest element for any $1 \leq k \leq n$

- Special cases: min $k = 1$, max $k = n$:
    - Linear time, $O(n)$
- What about **median** $k = \lfloor n + 1 \rfloor / 2$?
    - Sorting: $O(n \log n)$
    - Binary heap: $O(n \log k)$

Question. Can we do it in $O(n)$?

- **Surprisingly yes.**
- Selection is easier than sorting.

Selection: Problem Statement

Example. Take this array of size 10:

$$A = 12 \,|\, 2 \,|\, 4 \,|\, 5 \,|\, 3 \,|\, 1 \,|\, 10 \,|\, 7 \,|\, 9 \,|\, 8$$

Suppose we want to find 4th smallest element

- First, take any pivot p from $A[1, \ldots n]$
- If p is the 4th smallest element, return it
- Else, we partition A around p and recurse

Selection Algorithm: Idea

Select (A, k):

If $|A| = 1$: return $A[1]$

Else:

- Choose a pivot $p \leftarrow A[1,\ldots,n]$; let r be the rank of p
- $r, A_{<p}, A_{>p} \leftarrow$ Partition((A, p))
- If $k == r$, return p
- Else:
  - If $k < r$: Select $(A_{<p}, k)$
  - Else: Select $(A_{>p}, k - r)$

Selection: Problem Statement

Example. Take this array of size 10:

$A = 12\,|\,2\,|\,4\,|\,5\,|\,3\,|\,1\,|\,10\,|\,7\,|\,9\,|\,8$

Suppose we want to find 4th smallest element

- Choose pivot 8
- What is its rank?
  - Rank 7
- So let's find all of the smaller elements of A:
  - $A' = 2\,|\,4\,|\,5\,|\,3\,|\,1\,|\,7$
- Want to find the element of rank 4 in this new array

Selection: Problem Statement

Example. Take this array of size 10:

$$A = 12\,|\,2\,|\,4\,|\,5\,|\,3\,|\,1\,|\,10\,|\,7\,|\,9\,|\,8$$

Suppose we want to find 4th smallest element

- Choose as pivot 3
- What is its rank?
  - Rank 3
- So let's find all of the **larger** elements of A:
  - $A' = 12\,|\,4\,|\,5\,|\,10\,|\,7\,|\,9\,|\,8$
- Want to find the element of rank $4 - 3 = 1$ in this new array

When is this method good?

- If we guess the pivot right! (but we can't always do that)
- If we partition the array pretty evenly (the pivot is close to the middle)
  - Let's say our pivot is not in the first or last $3/10$ths of the array
  - What is our recurrence?
  - $T(n) \leq T(7n/10) + O(n)$
  - $T(n) = O(n)$

Our high-level goal

- Find a pivot that's close to the median—has a rank between $3n/10$ and $7n/10$, in time $O(n)$

- But the array is unsorted? How do we do that?

- Want to *always* be successful

Finding an Approximate Median

- Divide the array of size n into $\lceil n/5 \rceil$ groups of 5 elements (ignore leftovers)
- Find median of each group

29	10	38	37	2	55	18	24	34	35	36
22	44	52	11	53	12	13	43	20	4	27
28	23	6	26	40	19	1	46	31	49	8
14	9	5	3	54	30	48	47	32	51	21
45	39	50	15	25	16	41	17	22	7	

n = 54

Finding an Approximate Median

- Divide the array of size n into $\lceil n/5 \rceil$ groups of 5 elements (ignore leftovers)
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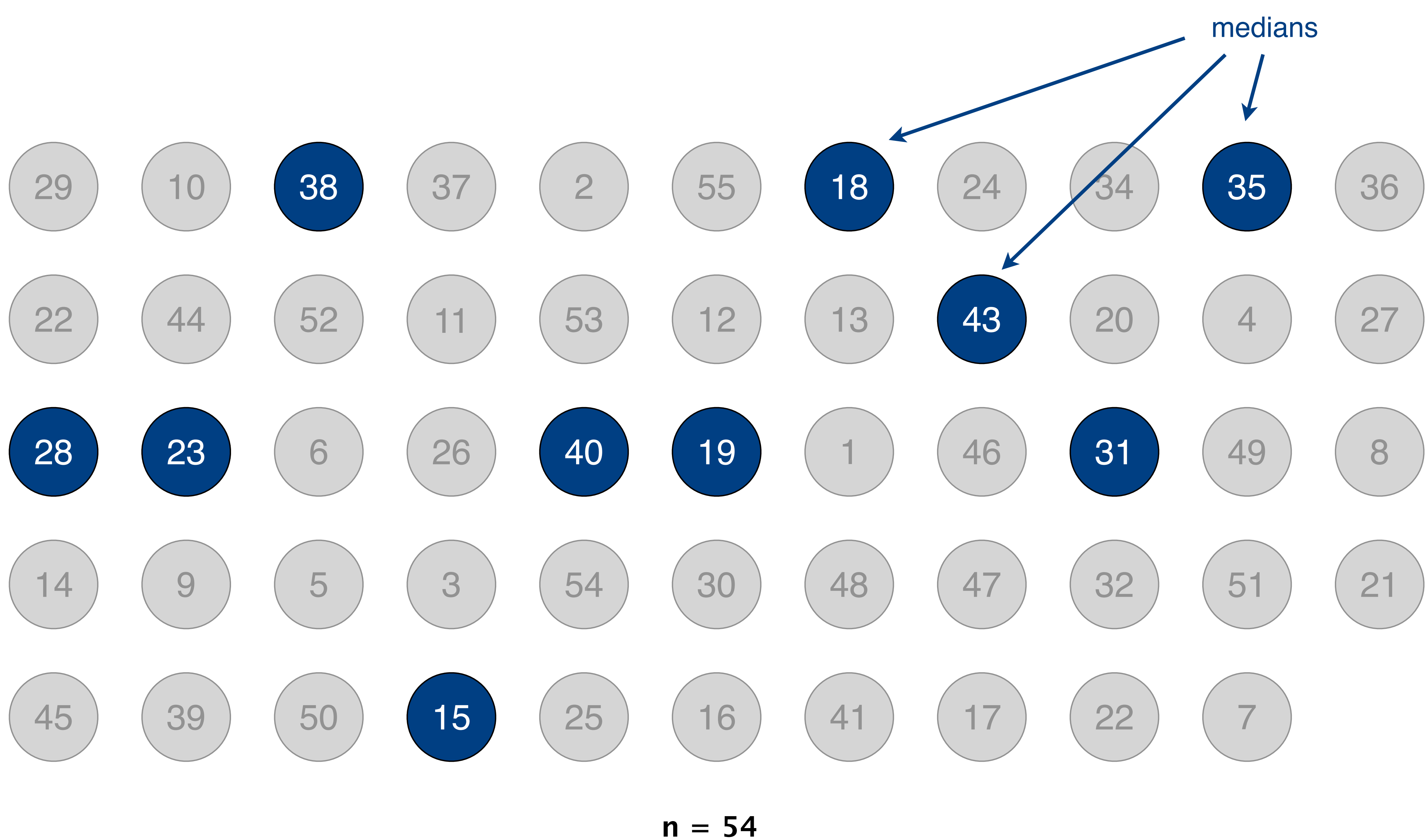
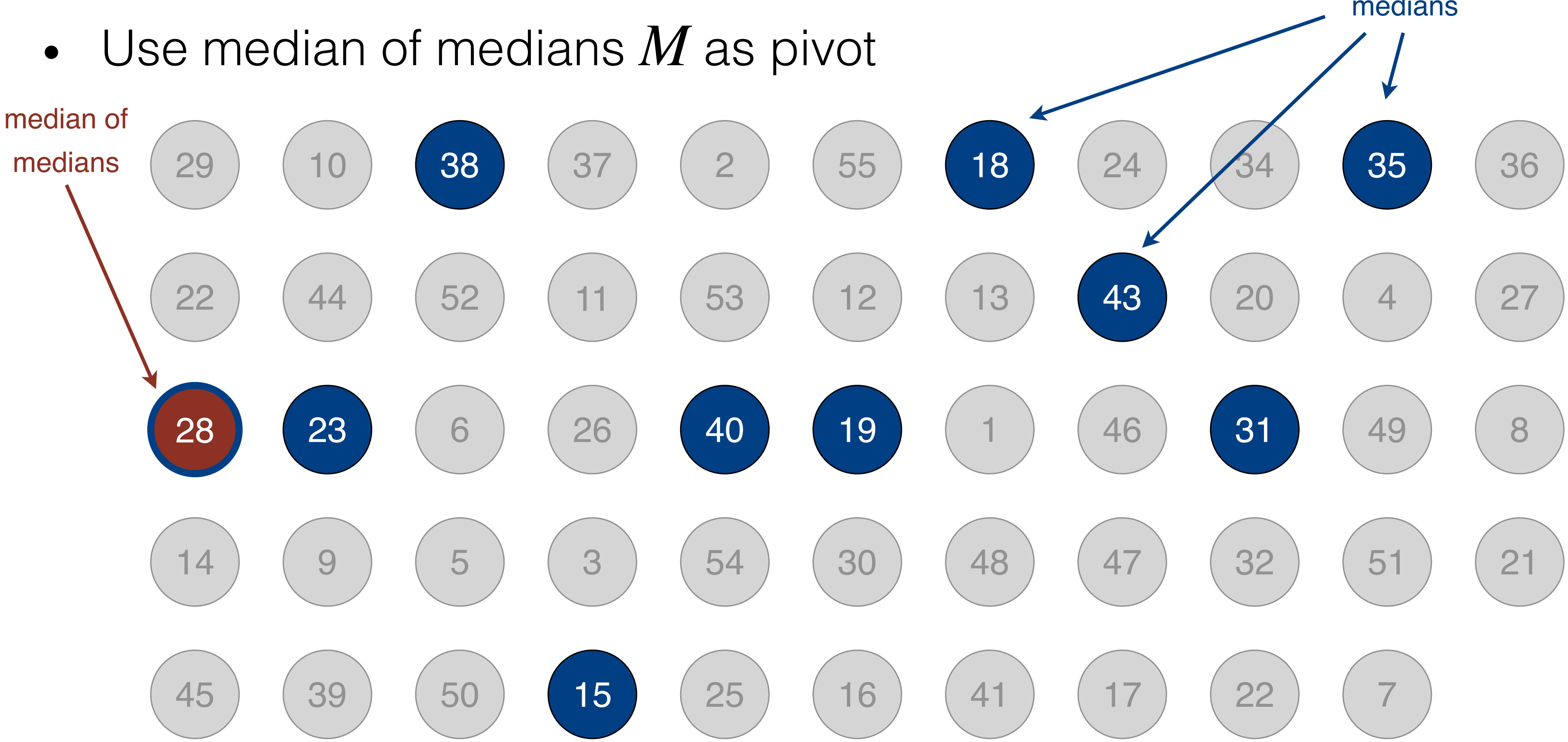


Finding an Approximate Median

- Divide the array of size n into $\lceil n/5 \rceil$ groups of 5 elements (ignore leftovers)
- Find median of each group
- Find $M \leftarrow$ median of $\lceil n/5 \rceil$ medians recursively
- Use median of medians M as pivot



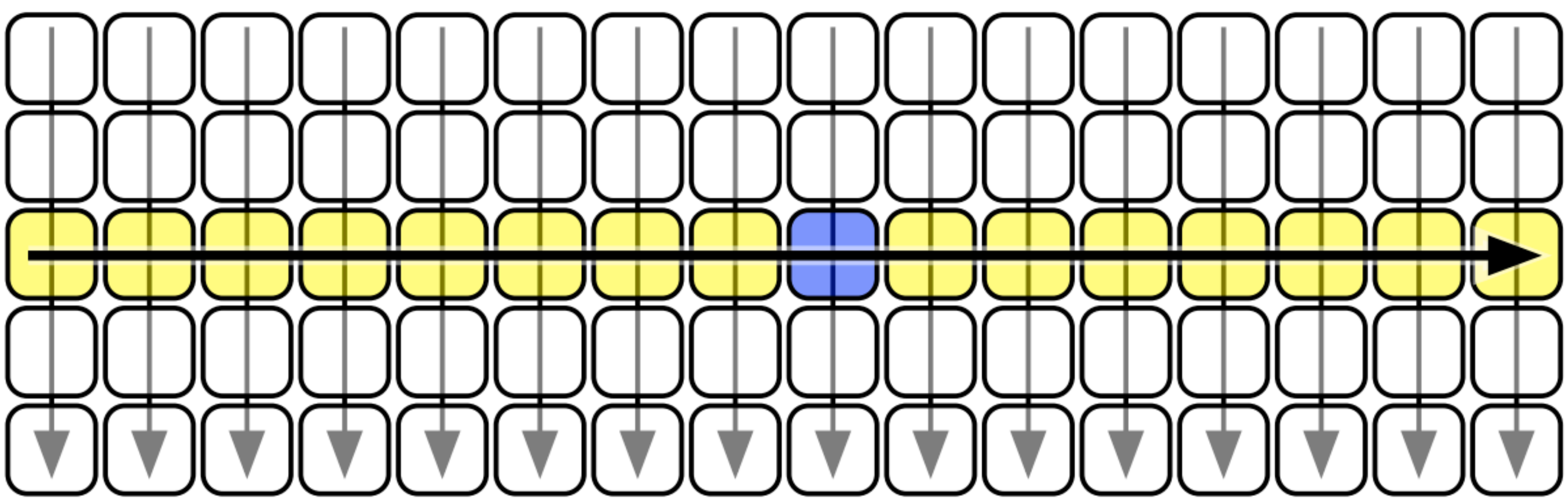
n = 54

What did we gain?

- How can I show that the median of medians is “close to the center” of the array?
- What elements can I say, for sure, are $\leq$ the median of medians?
  - The smaller half of the medians
  - $n/10$ elements
- Any other elements?
  - Another 2 elements in each median’s list

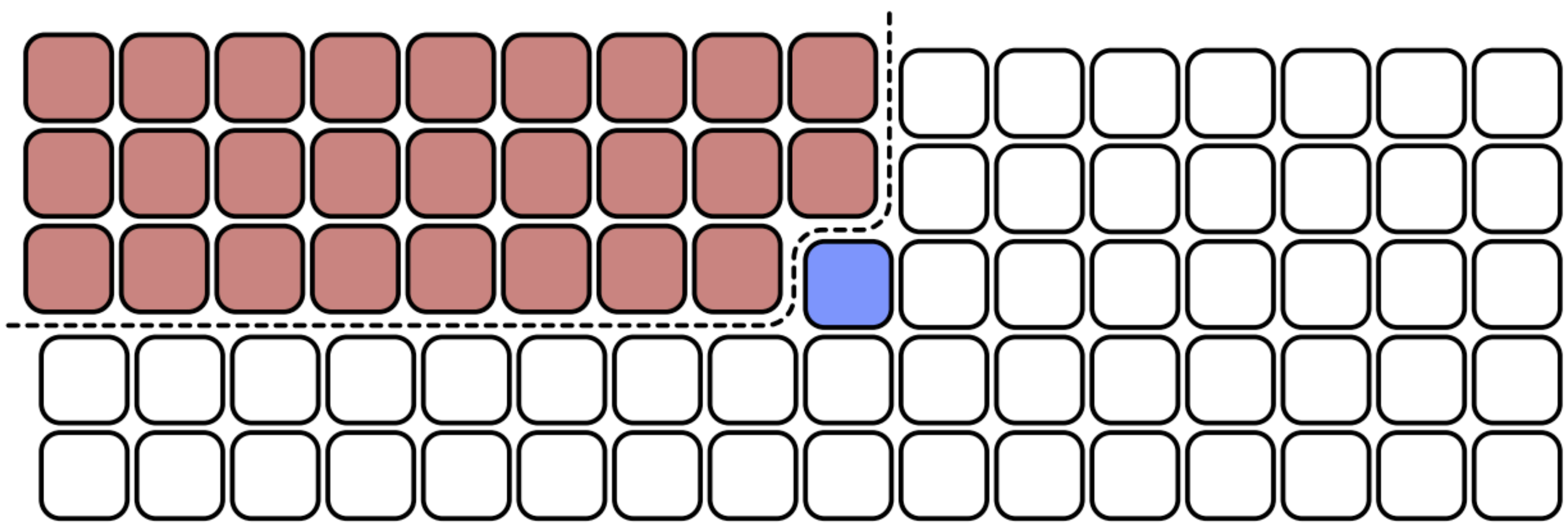
Visualizing MoM

- In the 5 x n/5 grid, each column represents five consecutive elements
- **Imagine** each column is sorted top down
- **Imagine** the columns as a whole are sorted left-right
    - We don't actually do this!
- MoM is the element closest to center of grid



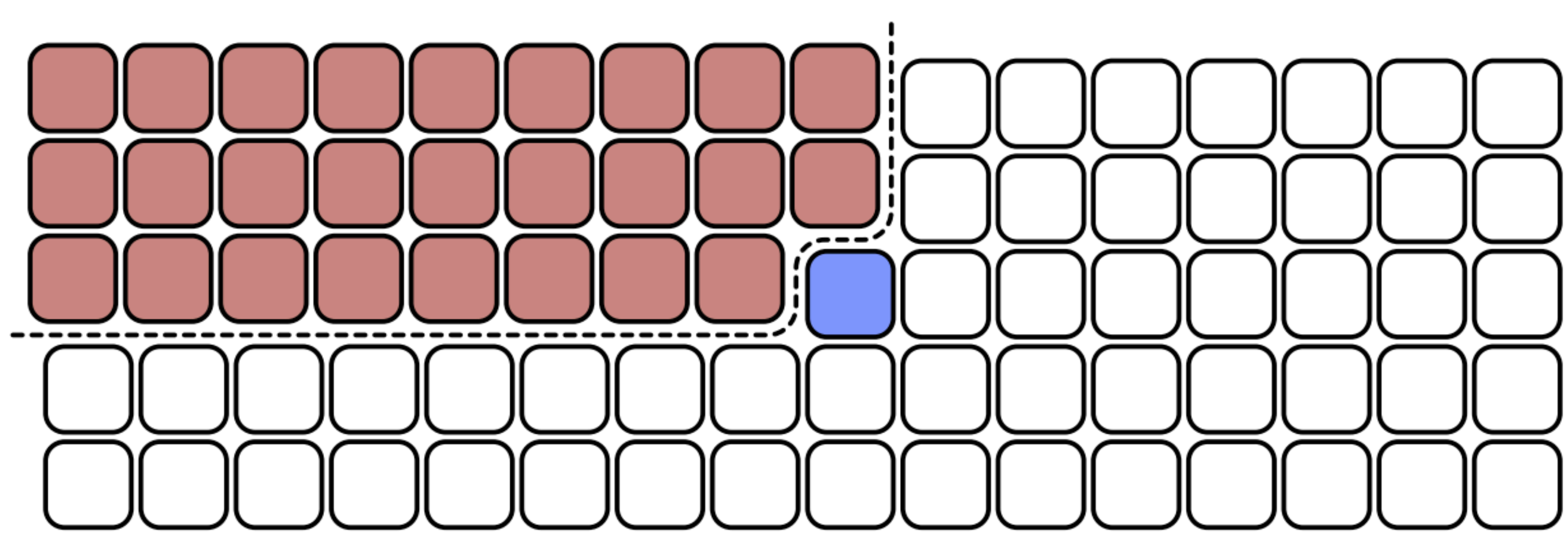
Visualizing MoM

- Red cells (at least $3n/10$) are smaller than M



Visualizing MoM

- Red cells (at least $3n/10$) in size are smaller than M
- If we are looking for an element larger than M, we can throw these out, before recursing
- Symmetrically, we can throw out $3n/10$ elements larger than M if looking for a smaller element
- Thus, the recursive problem size is at most $7n/10$



How Good is Median of Medians

Claim. Median of medians M is a good pivot, that is, at least $3/10$th of the elements are $\geq M$ and at least $3/10$th of the elements are $\leq M$.

Proof.

- Let $g = \lceil n/5 \rceil$ be the size of each group.
- M is the median of g medians
  - So $M \geq g/2$ of the group medians
  - Each median is greater than 2 elements in its group
  - Thus $M \geq 3g/2 = 3n/10$ elements
- Symmetrically, $M \leq 3n/10$ elements. ∎

Median of Medians Subroutine

- MoM(A, n):
  - If $n == 1$: return $A[1]$
  - Else:
    - Divide A into $\lceil n/5 \rceil$ groups
    - Compute median of each group
    - $A' \leftarrow$ group medians
    - Mom(A', $\lceil n/5 \rceil$)

$T(n/5) + O(n)$

Linear time Selection

Select (A, k):

If $|A| = 1$: return $A[1]$; else:

- Call median of medians to find a good pivot

  $p \leftarrow \text{MoM}(A, n)$; $n = |A|$

$T(n/5) + O(n)$

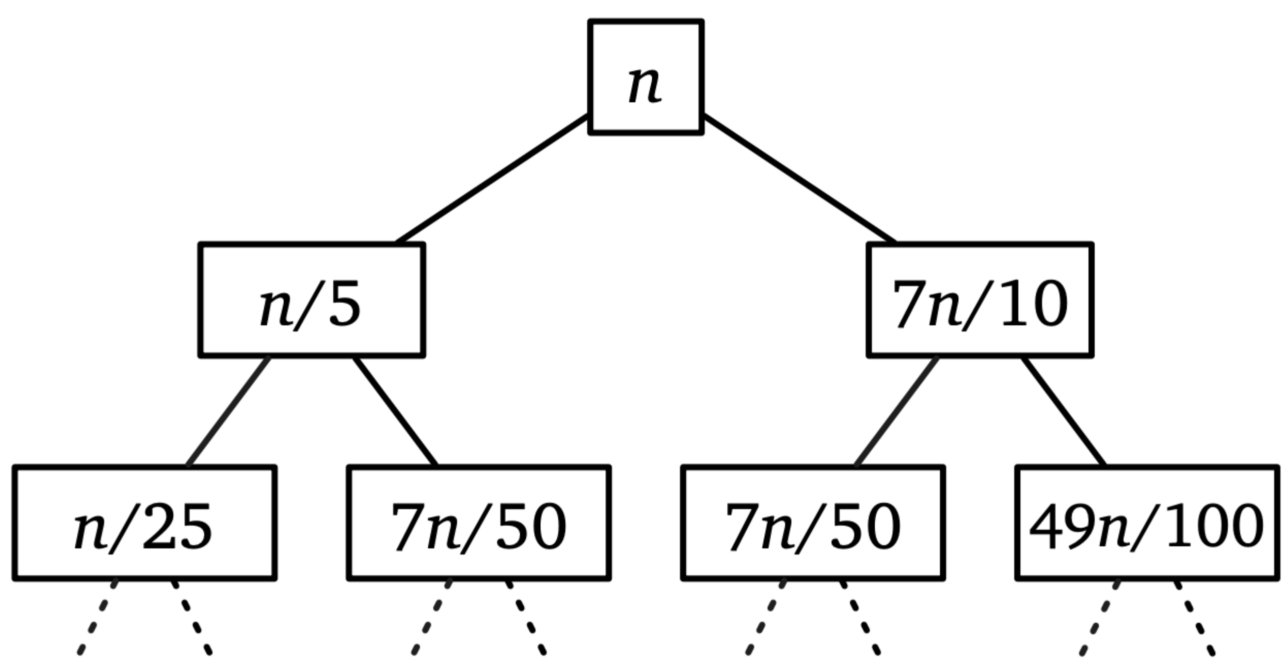
- $r, A_{<p}, A_{>p} \leftarrow$ Partition((A, p))
- If $k == r$, return p
- Else:
  - If $k < r$: Select $(A_{<p}, k)$
  - Else: Select $(A_{>p}, k - r)$

Larger subproblem has size $\leq 7n/10$

Overall: $T(n) = T(n/5) + T(7n/10) + O(n)$

Selection Recurrence

- Okay, so we have a good pivot
- We are still doing two recursive calls
    - $T(n) \leq T(n/5) + T(7n/10) + O(n)$
- Key: total work at each level still goes down!
- Decaying series gives us : $T(n) = O(n)$



Why the Magic Number 5?

- What was so special about 5 in our algorithm?
- It is the smallest odd number that works!
    - (Even numbers are problematic for medians)
- Let us analyze the recurrence with groups of size 3
    - $T(n) \leq T(n/3) + T(2n/3) + O(n)$
    - Work is equal at each level of the tree!
    - $T(n) = \Theta(n \log n)$

Theory vs Practice

- $O(n)$-time selection by [Blum–Floyd–Pratt–Rivest–Tarjan 1973]
  - Does $\leq 5.4305n$ compares
- Upper bound:
  - [Dor–Zwick 1995] $\leq 2.95n$ compares
- Lower bound:
  - [Dor–Zwick 1999] $\geq (2 + 2^{-80})n$ compares.
- Constants are still too large for practice
- Random pivot works well in most cases!
  - We may analyze this when we do randomized algorithms

Acknowledgments

- Some of the material in these slides are taken from
    - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
    - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)