Shortest Path Problem

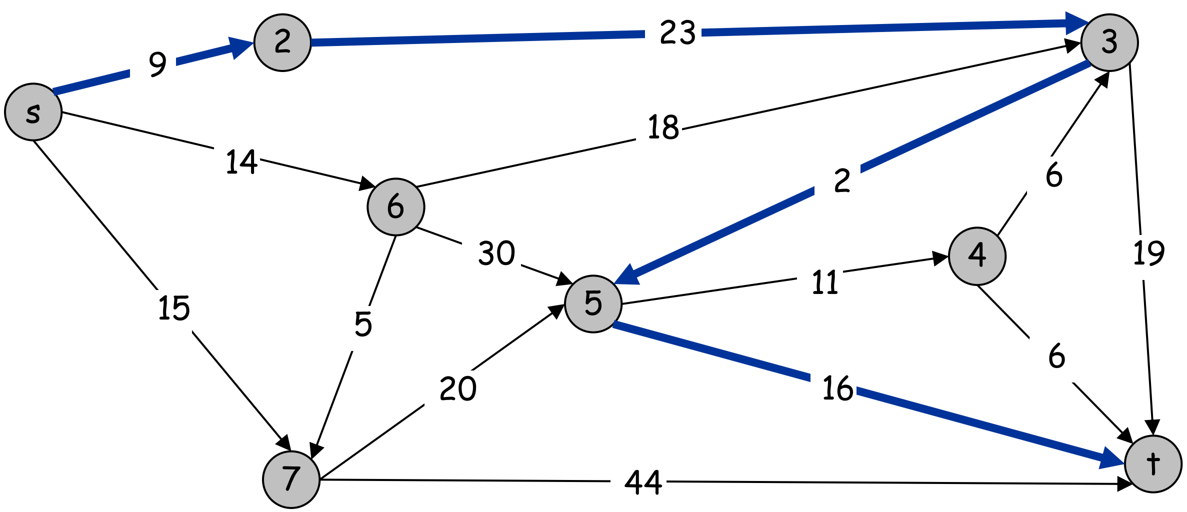
Shortest path in a network.

- Directed graph $G = (V, E)$.
- Source s, destination t.
- Length ℓ_e = length of edge e.

cost of path = sum of edge costs in path

Shortest path problem: find shortest (directed) path from s to t.

Single source shortest path problem: find shortest directed path from s to every node in V



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 48.

Dijkstra's Algorithm

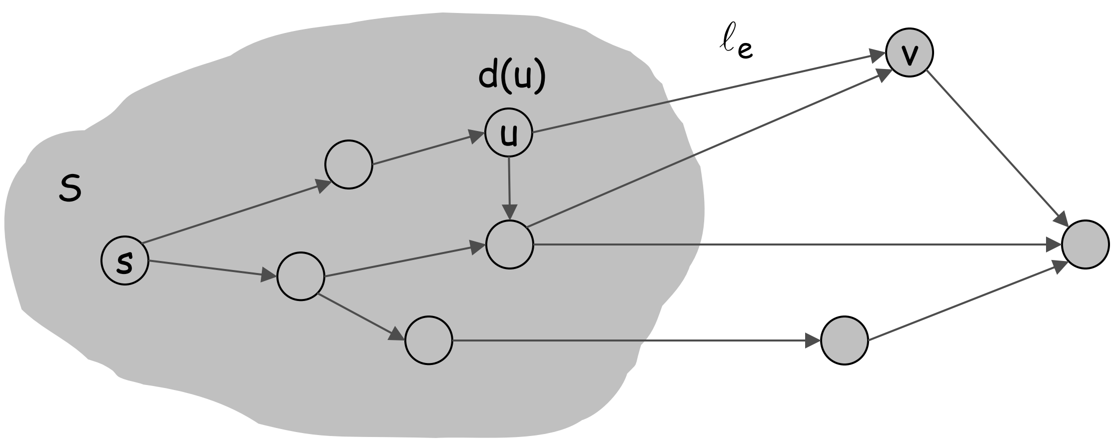
Dijkstra's algorithm.

- Maintain a set S of explored nodes for which we have determined the shortest path distance $d(u)$ from s to u.
- Initialize $S = \{ s \}$, $d(s) = 0$.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + l_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set $d(v) = \pi(v)$.



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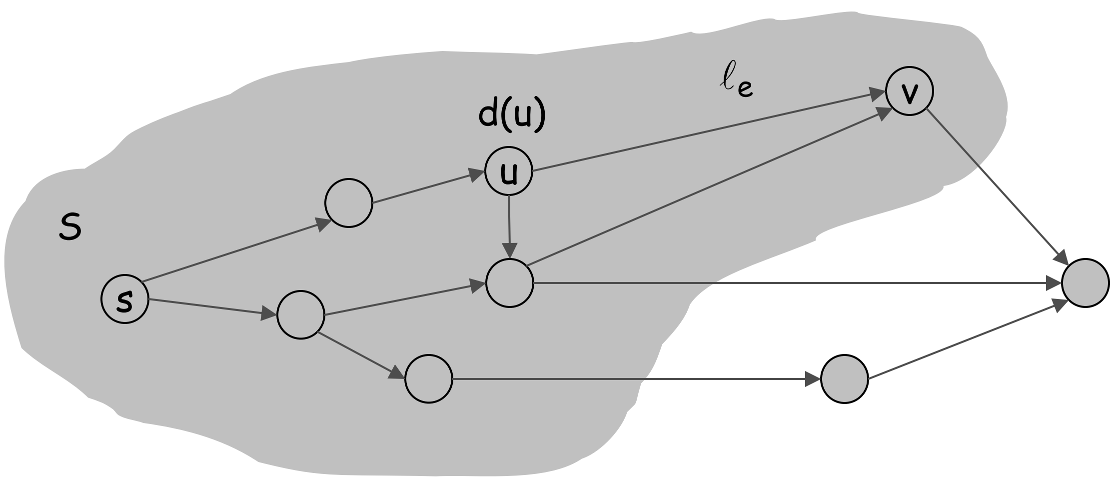


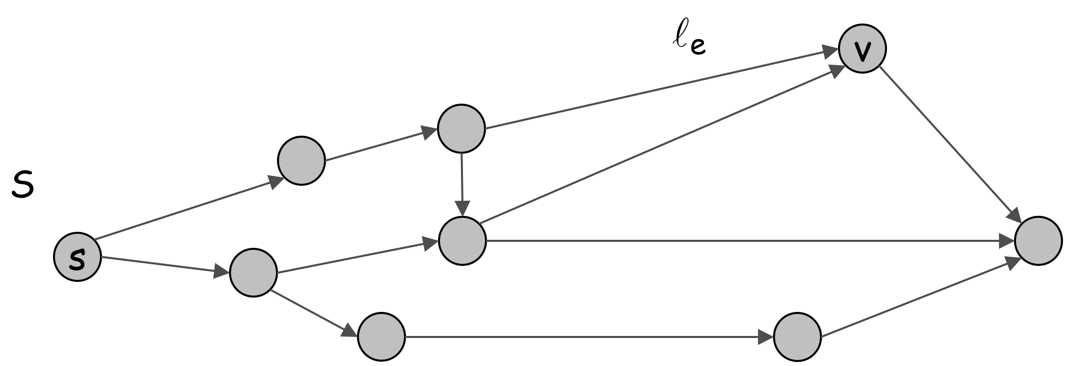
Dijkstra's Algorithm

Dijkstra's algorithm.

- Dijkstra's algorithm is a greedy algorithm.
  - What defines a "step" towards our goal?
  - What is our optimization criteria at each step?

- The result is a globally optimal solution to the SSSP problem!

- How to implement?



Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e$.

- Next node to explore = node with minimum $\pi(v)$.
- When exploring v, for each outgoing edge e = (v, w), update

$$\pi(w) = \min \{ \pi(w), \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.

	Priority Queue	
PQ Operation	Array	Binary heap
Insert	n	log n
ExtractMin	n	log n
ChangeKey	1	log n
IsEmpty	1	1
Total	n^2	m log n

Dijkstra's Algorithm Pseudocode

```
Dijkstra(G, s):
    let T←({s}, ∅)
    let PQ be an empty priority queue

    for each neighbor v of s, add edge (s,v) to PQ with priority l(e)
    while T doesn't have all vertices of G and PQ is non-empty:
            repeat {
                e←PQ.removeMin() // skip edges with both ends in T
            } until PQ is empty or e=(u,v) for u∈T, v ∉ T
            if e=(u,v) for u∈T, v ∉ T
                add e (and v) to T
                for each neighbor w of v
                    add edge (v,w) to PQ with weight/key d(s,v) + l(v,w)
```

Dijkstra's Algorithm: Proof of Correctness

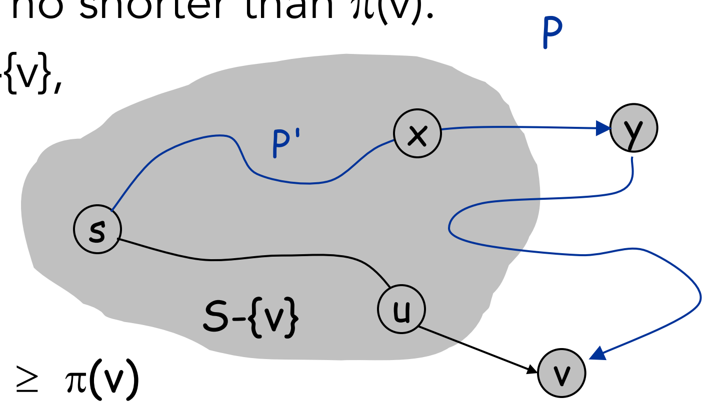
Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest s-u path.

Pf. (by induction on $|S|$)

Base case: $|S| = 1$ and $d(s)=0$, which is true.

Inductive hypothesis: Assume true for $|S| = k \le n$. Consider $|S|=k+1$

- Let v be last node added to S, and let u-v be the chosen edge.
- By inductive hypothesis, all nodes in S-{v} have correct shortest path dis.
- Claim: the s-u path plus (u, v) is an s-v path of shortest length $\pi(v)$.
  - Consider any s-v path P. We'll see that it's no shorter than $\pi(v)$.
  - Let x-y be the first edge in P that leaves S-{v}, and let P' be the subpath to x.



$$\underbrace{\ell(P) \ge \ell(P') + \ell(x,y)}_{\text{nonnegative weights}} \;\; \underbrace{\ge d(x) + \ell(x,y)}_{\text{inductive hypothesis}} \;\; \underbrace{\ge \pi(y)}_{\text{defn of } \pi(y)} \;\; \underbrace{\ge \pi(v)}_{\text{Dijkstra chose v instead of y}}$$