Largest Subinterval Sum & Asymptotic Analysis

Outline

- Look at a fun problem (Largest Subinterval Sum)
- Iteratively develop more efficient solutions
  - Prove some things to help us get there
- Take a step back and state precisely what we mean by efficiency
- Practice some asymptotic analysis
- Review helpful log manipulation tricks

Largest Subinterval Sum

INPUT: An array A of n integers (1-indexed)

OUTPUT: The largest sum of any subinterval. The empty interval (which we will represent as $NULL$) has sum 0.

Example 1: Consider the array $(10, 20, -50, \underline{40})$

Subinterval [1, 1] = 10

Subinterval [1,4] = 10+20-50+40 = 20

Subinterval [2, 3] = 20-50 = -30

The largest subinterval sum is 40, corresponding to $[4,4]$

Largest Subinterval Sum

INPUT: An array A of n integers (1-indexed)

OUTPUT: The largest sum of any subinterval. The empty interval (which we will represent as $NULL$) has sum 0.

Example 2: Consider the array $(-2, \underline{3, -2, 4, -1, 8}, -20)$

The largest subinterval sum is 12, corresponding to $[2,6]$

Largest Subinterval Sum

INPUT: An array A of n integers (1-indexed)

OUTPUT: The largest sum of any subinterval. The empty interval (which we will represent as $NULL$) has sum 0.

Question: Is this problem interesting when the array's integers are all positive?

No! Then the answer is always the entire interval…

Developing an Algorithm

Algorithm with $O(n^3)$ Steps

- Let's start with an algorithm that corresponds directly to the problem definition:
  - We are looking for the largest sum of any sub-interval
    - How many total sub-intervals are there?
      - $\binom{n}{2}$ which is $\frac{n(n+1)}{2} = O(n^2)$
    - How long does it take to sum a sub-interval?
      - $O(n)$ (in the worst case, must sum entire array)

This brute-force algorithm takes $O(n^3)$ steps

LargestSum(A):

$largest \leftarrow 0$

for $i \leftarrow 1 \ldots n$

for $j \leftarrow i \ldots n$

$sum \leftarrow 0$

for $k \leftarrow i \ldots j$

$sum \leftarrow sum + A[k]$

$largest \leftarrow \max(sum, largest)$

return $largest$

Try walking through LargestSum(A) on a small example, like $A = (10,20,-50,40)$

Algorithm with $O(n^2)$ Steps

- The last algorithm repeated a lot of work. How?
  - If A had 7 integers, interval $[2,7]$ computed $[2,2]$, $[2,3]$, $[2,4]$, and so on…
    - Can we avoid this repeated work?

Idea: Compute and reuse a *Partial Sum* table

$$PS(j) = \sum_{i=1}^{j} A(i)$$

Algorithm with $O(n^2)$ Steps

Claim: We can use PS to compute the sum of any interval (i, j) in $O(1)$ time. How?

A		-2	3	-2	4	-1	8	-20
PS	0	-2	1	-1	3	2	10	-10

$PS[i]$ contains sum of all integers "up until $A[i]$", with a 0 for the empty array.

$$PS(j) = \sum_{i=1}^{j} A(i)$$

Algorithm with $O(n^2)$ Steps

Example: How to compute $A(3,6)$?

A								
		-2	3	-2	4	-1	8	-20

PS								
	0	-2	1	-1	3	2	10	-10

$PS[2]$ is everything before $A[3]$

$PS[6]$ is everything up to $A[6]$

$$PS(j) = \sum^{j} A(i)$$

Subtract $PS[j] - PS[i-1]$

LargestSum(A):

$PS \leftarrow partial\_sums(A)$ // we can construct this in O(n) time

$largest \leftarrow 0$

for $i \leftarrow 1 \ldots n$

for $j \leftarrow i \ldots n$

$largest \leftarrow \max(largest, PS[j] - PS[i-1])$

return $largest$

$O(n^2)$ iterations

Each iteration performs $O(1)$ work

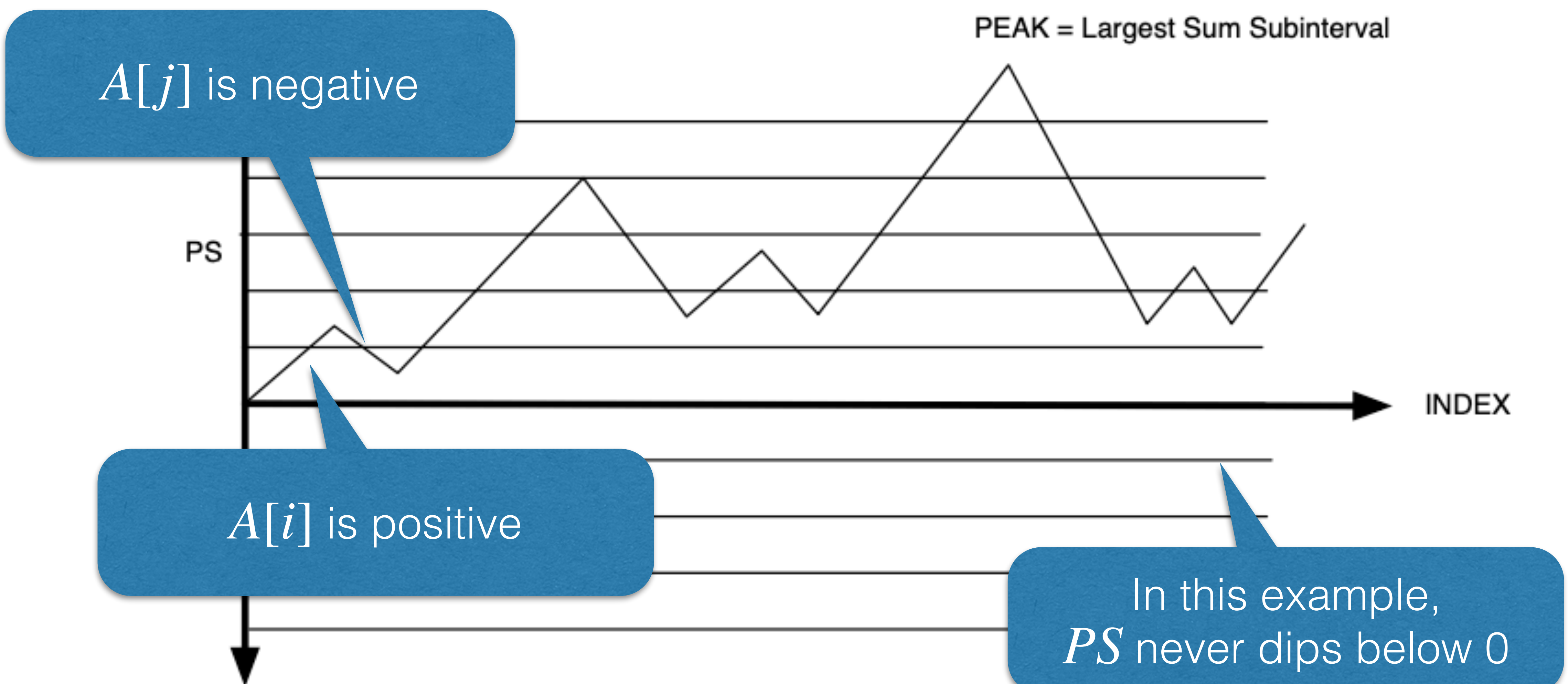
Total cost: $O(n^2)$

Can We Do Even Better?

Algorithm with $O(n)$ Steps

Let $PS(j) = \sum_{i=1}^{j} A[i]$ give the partial sum of the first j integer values of A.

Let's visualize an example $PS(j)$



Algorithm with $O(n)$ Steps

Observation 1: If $PS(j) \geq 0$ for all $1 \leq j \leq n$ then the largest sum subinterval is the interval $[1,k]$ where k maximizes $PS(k)$.

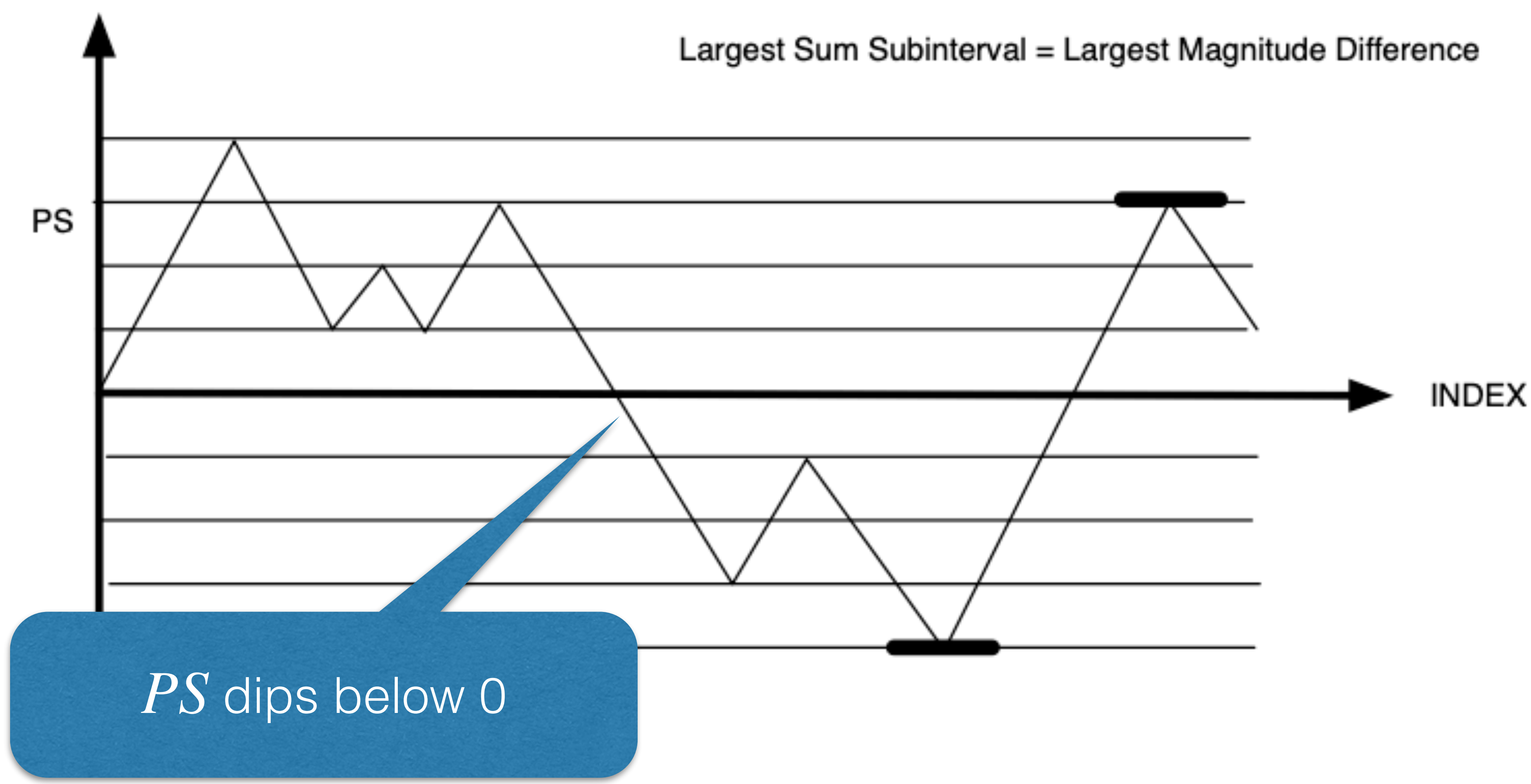
Proof. The proof is by contradiction.

Suppose $[1,k]$ did not give the largest sum. Then there is some other interval $[u, v]$ that has a larger sum. But shifting u to 1 cannot decrease the sum (since we would then be subtracting out 0), and shifting v to k cannot decrease the sum (since k maximizes $PS(k)$). Thus $[u, v]$ cannot be an interval with a larger sum.

Algorithm with $O(n)$ Steps

Let $PS(j) = \sum_{i=1}^{j} A[i])$ give the partial sum of the first j integer values of A.

Let's visualize a second example $PS(j)$:



Algorithm with $O(n)$ Steps

Observation 2: When $PS(j)$ falls below 0 for the first time, then the largest sum subinterval never includes j—it falls on one side or the other. That is, when $PS(j)$ falls below 0 for the first time, the problem essentially "resets" with $PS(j)$ being "the new 0".

Proof. The proof is by contradiction.

Suppose the largest sum subinterval $[u, v]$ contains the first point j where the partial sum drops below 0. Notice that $[u, j]$ corresponds to a negative sum. The interval $[j+1, v]$ must be larger than $[u, v]$ since we are subtracting out a negative sum. This is a contradiction.

LargestSum(A):

$sum, largest \leftarrow 0$

for $i \leftarrow 1 \ldots n$

$\quad sum \leftarrow \max(sum + A[i], 0)$

$\quad largest \leftarrow \max(sum, largest)$

return $largest$

This $O(n)$ algorithm follows from our previous two observations.

- We only need to worry about sums corresponding to intervals where i is a new "0-point" for the partial sum and j maximizes the partial sum
- Going back to our visualization, we are calculating the largest difference between some valley and a subsequent peak

Reflecting on our Algorithms

We proposed and analyzed three algorithms that find the largest subinterval sum problem

- All three algorithms are correct
- When given the same input, not all three algorithms will complete in the same number of steps

The type of analysis we did is called asymptotic analysis, and it's something we'll do throughout the rest of this course

Analysis and Asymptotics

Why should we examine problems analytically?

- Analysis is independent of the algorithm's implementation, the language the program is written in, and the hardware on which the program is run
- Theoretical efficiency almost always implies a path towards practical efficiency
  - When there is a mismatch between a theoretical model's predictions and the observed performance, there is an interesting systems problem to be solved!

My research group relies on this!

Analysis and Asymptotics

Why use worst-case analysis?

- Worst-case is a *real* guarantee.
- Worst-case captures efficiency reasonably well in practice. Exceptions are rare (e.g., Quicksort) and interesting.
- Average case is hard to quantify—we often don't know the true distribution of inputs, so what are we analyzing the average of?

Analysis and Asymptotics

- *What does efficient actually mean?*
  - We will say an algorithm is efficient if it runs in time that is polynomial in the size of the input
  - Practical efficiency probably maxes out somewhere between $O(n \log n)$ and $O(n^3)$, depending on the context
  - ***Not*** brute force!

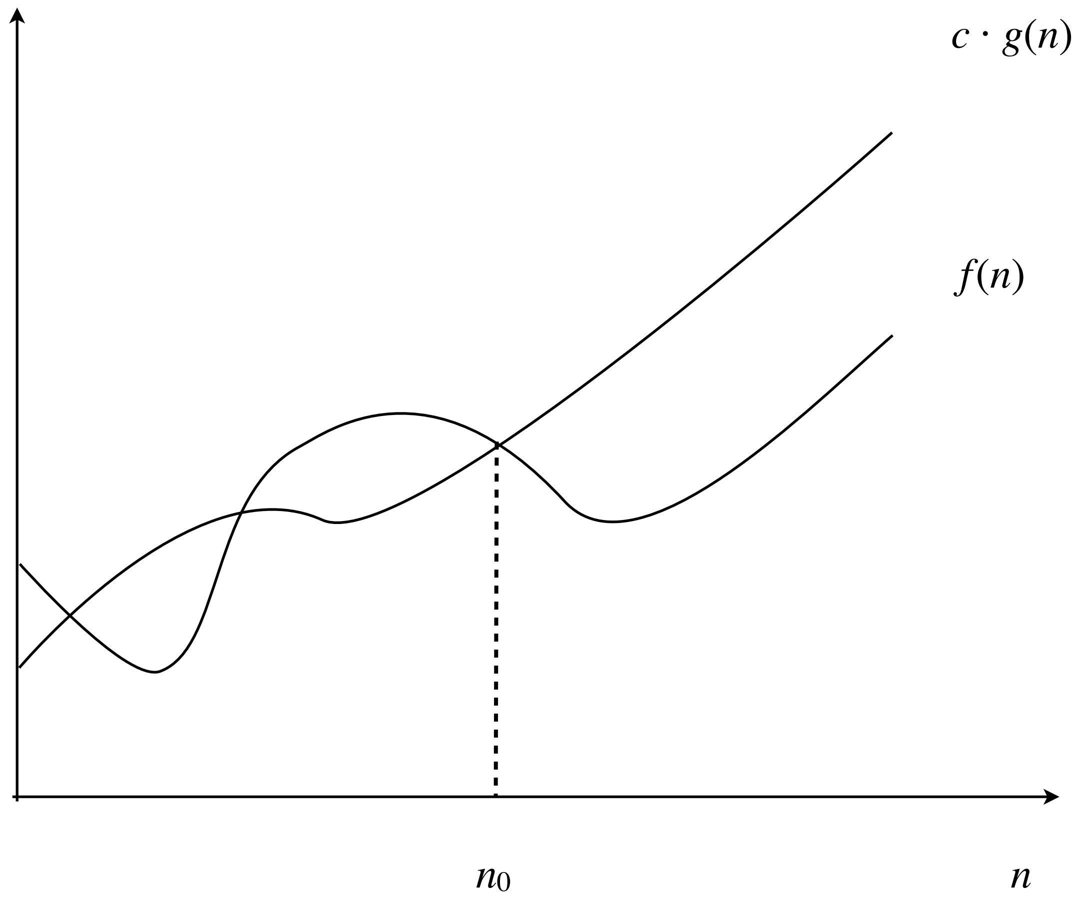
Analysis and Asymptotics

- *Why use asymptotic analysis?*
  - Precise bounds are difficult to calculate
  - Precise runtime is dependent on external factors, often including things we don't consider or can't control (hardware, OS environment, compiler, …)
  - We often want to *compare* algorithms, and equivalency up to constant factors is often the right level of detail to have those conversations
    - Once we pick an efficient algorithm, we can optimize the "practical considerations" during its implementation

Asymptotic Analysis

Big-O

Definition (Asymptotic upper bounds): $f(n)$ is $O(g(n))$ if and only if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \leq c \cdot g(n)$



Big-O

Definition (Asymptotic upper bounds): $f(n)$ is $O(g(n))$ if and only if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \leq c \cdot g(n)$

Example:

$$\begin{aligned} f(n) &= 3n^2 + 17n + 8 \\ &\leq 3n^2 + 17n^2 + 8n^2 \quad \text{For } n \geq 1 \\ &= 28n^2 \end{aligned}$$

Choosing $c = 28$ and $n_0 = 1$ means $f(n)$ is $O(n^2)$

Concept Check

Let $f(n) = 3n^2 + 17n\log_2 n + 1000$. Which of the following are true?

A. $f(n)$ is $O(n^2)$.

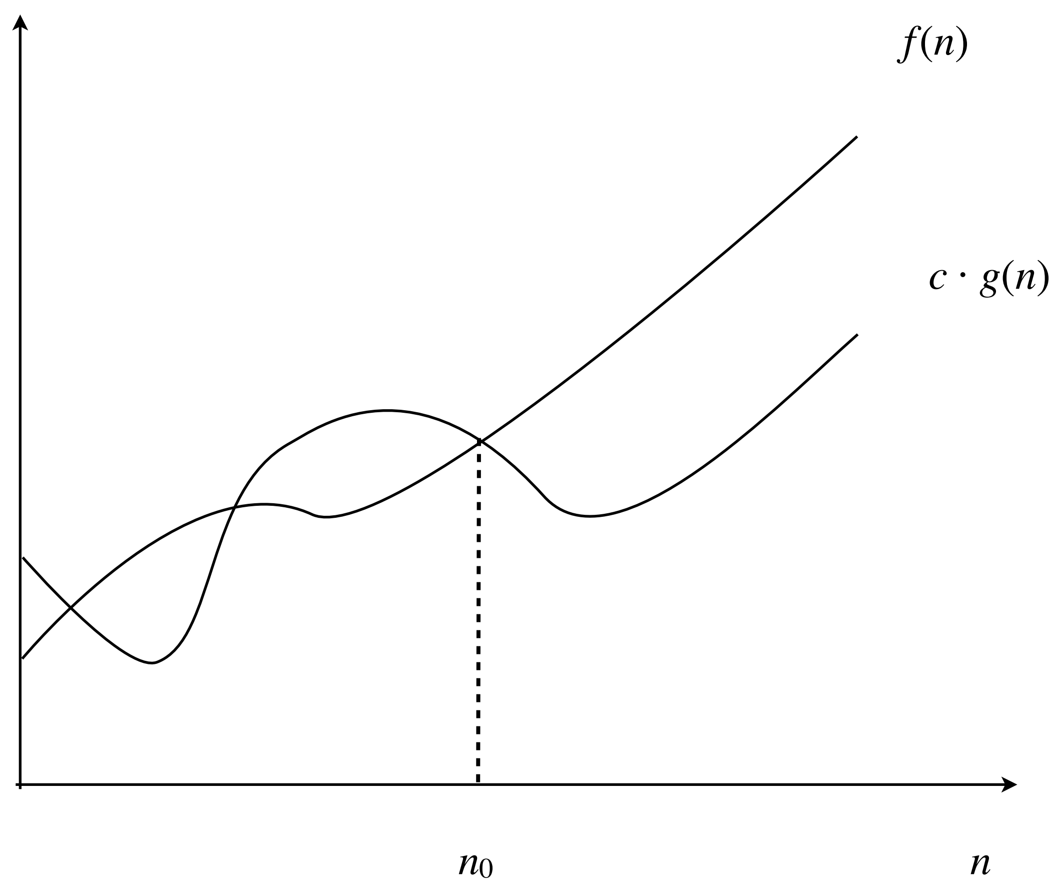
B. $f(n)$ is $O(n^3)$.

C. Both A and B.

D. Neither A nor B.

Big-Omega

Definition (Asymptotic lower bounds): $f(n)$ is $\Omega(g(n))$ if and only if there exists constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq c \cdot g(n)$



Big-Omega

Definition (Asymptotic lower bounds): $f(n)$ is $\Omega(g(n))$ if and only if there exists constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq c \cdot g(n)$

Example:

$$f(n) = 3n^2 + 17n + 8$$

$$\geq 3n^2 \qquad \text{For } n \geq 0$$

Choosing $c = 1$ and $n_0 = 0$ means $f(n)$ is $\Omega(n^2)$

Concept Check

Let $f(n) = 3n^2 + 17n \log_2 n + 1000$. Which of the following are true?

A. $f(n)$ is $\Omega(n^2)$.

B. $f(n)$ is $\Omega(n^3)$.

C. Both A and B.

D. Neither A nor B.

Big-Theta

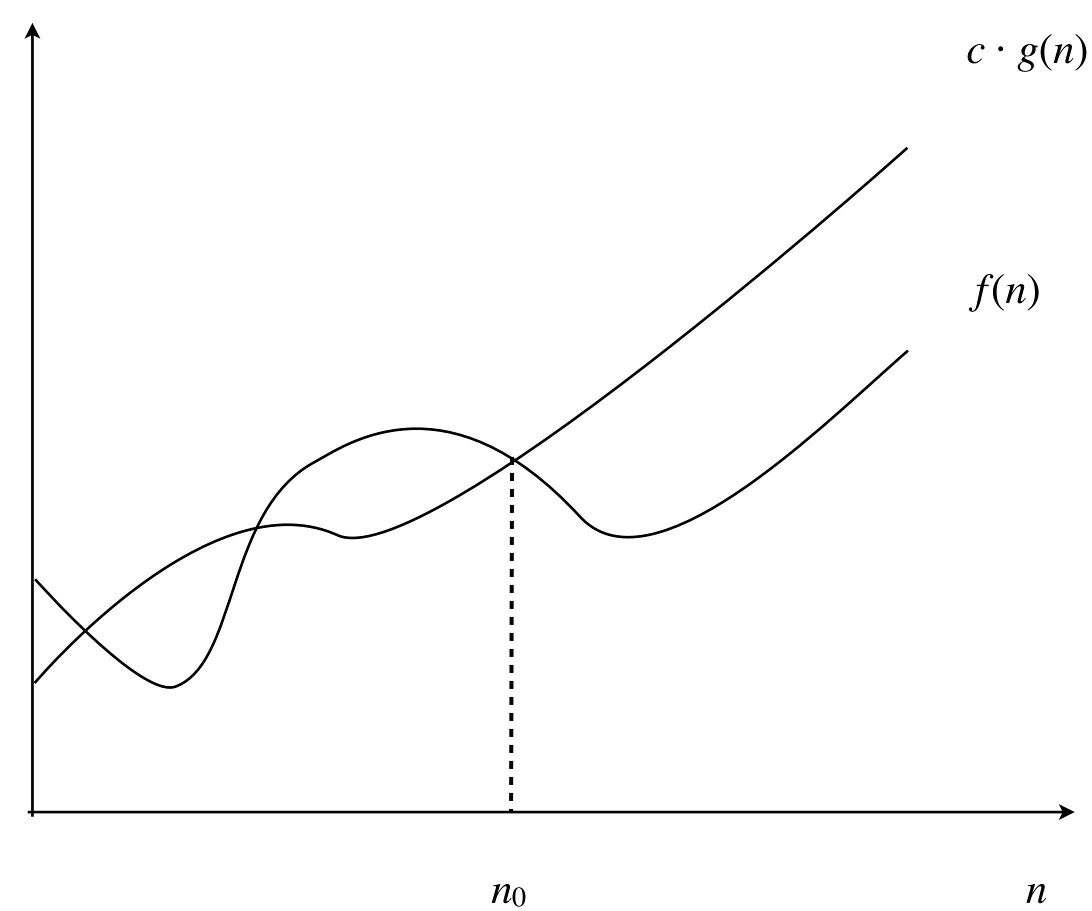
Definition (Asymptotic tight bounds): $f(n)$ is $\Theta(g(n))$ if and only if $f(n)$ is $O(g(n))$ and $\Omega(g(n))$

Equivalently, if there exist constants $c_1 > 0$, $c_2 > 0$, and $n_0 \geq 0$ such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.

Ideally, we'd strive for a "tight" bounds whenever we can!

Big Oh- Notational Abuses

- $O(g(n))$ is actually a set of functions, but the CS community writes $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$
- For example
  - $f_1(n) = O(n \log n) = O(n^2)$
  - $f_2(n) = O(3n^2 + n) = O(n^2)$
  - But $f_1(n) \neq f_2(n)$
- Okay to abuse notation in this way



Growth of Functions

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Playing with Logs: Properties

- In this class, $\log n$ means $\log_2 n$, $\ln n = \log_e n$
- Constant base doesn't matter for big-O:

$$\log_b(n) = \frac{\log n}{\log b} = O(\log n)$$

- Useful properties of logs:
  - $\log(n^m) = m \log n$
  - $\log(ab) = \log a + \log b$
  - $\log(a/b) = \log a - \log b$

$$a^{\log_a n} = n$$

We will use this a lot!

Exponents

$$n^a \cdot n^b = n^{a+b}$$

$$(n^a)^b = n^{ab}$$

Comparing Running Times

- When comparing two functions, helpful to simplify first
- Is $n^{1/\log n} = O(1)$?
- Is $\log \sqrt{4^n} = O(n^2)$?
- Is $n = O(2^{\log_4 n})$?

Comparing Running Times

- When comparing two functions, helpful to simplify first
- Is $n^{1/\log n} = O(1)$?
  - Simplify $n^{1/\log n} = (2^{\log n})^{1/\log n} = 2$: **True**
- Is $\log \sqrt{4^n} = O(n^2)$
  - Simplify $\log \sqrt{2^{2n}} = \log 2^n = n \log 2 = O(n)$: **True**
- Is $n = O(2^{\log_4 n})$?
  - Simplify $2^{\log_4 n} = 2^{\frac{\log_2 n}{log_2 4}} = 2^{(\log_2 n)/2} = 2^{\log_2 \sqrt{n}} = \sqrt{n}$: **False**

Tools for Comparing Asymptotics

- We can use limits to show asymptotic bounds
  - If $\lim_{n\to\infty} \frac{f(x)}{g(x)} = 0$, then $f(x) = O(g(x))$
  - If $\lim_{n\to\infty} \frac{f(x)}{g(x)} = c$ for some constant $0 < c < \infty$,

    then $f(x) = \Theta(g(x))$

Tools for Comparing Asymptotics

- Logs grow more slowly than any polynomial:
  - $\log_a n = O(n^b)$ for every $a > 1,\ b > 0$
- Exponentials grow faster than any polynomial:
  - $n^d = O(r^n)$ for every $d > 1,\ r > 0$
- Taking logs is often useful for comparing function growth
  - Since $\log x$ is a strictly increasing function for $x > 0$, $\log(f(n)) < \log(g(n))$ implies $f(n) < g(n)$
  - E.g. Compare $3^{\log n}$ vs 2^n
    - Taking log of both, we get: $\log n \log 3$ vs n

But **BEWARE**: when comparing logs, the constants matter!