Stable Matching & Asymptotic Analysis
Reminders

- Confirm that your Gradescope account is set up & you can see the assignment submission portal
  - Assignment 0 due **Wed, Feb 8 at 10 pm**
- Bill’s office hours:
  - (Today) 11-noon
  - (Tomorrow) 3-4:30 pm
  - (Wednesday) 1:30-3pm

TAs:
- I plan to post the TA help schedule this afternoon
  - Largely 6-10pm in TCL 206
Resources

LaTeX guides & Overleaf

- https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes

Other Topics? How do we feel about these:

- Induction
- “Key” Data structures (APIs: pseudocode/sketching & Big-O)
  - Lists & arrays, trees, heaps, graphs, hash tables
- Asymptotic analysis building blocks
- Sorting
- Counting and Probability
Matching Med-Students to Hospitals

**Input.** A set $H$ of $n$ hospitals, a set $S$ of $n$ students and their preferences (each hospital ranks each student, each student ranks each hospital)

$H = \{ \text{MA, NH, OH} \}$

$S = \{ \text{Aamir, Beth, Chris} \}$

Intuitively:
What features make a matching *good*?
What features makes a matching *bad*?
**Definition.** A matching $M$ is a set of ordered pairs $(h, s)$ where $h \in H$ and $s \in S$ such that

- Each hospital $h$ is in at most one pair in $M$
- Each student $s$ is in at most one pair in $M$

A matching $M$ is **perfect** if each hospital is matched to exactly one student and vice versa (i.e., $|M| = |H| = |S|$)

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**Unstable Pairs**

**Definition.** A perfect matching $M$ is **unstable** if there exists an unstable pair $(h, s) \in H \times S$, that is, both of the following are true:

- $h$ prefers $s$ to its current match in $M$, and
- $s$ prefers $h$ to its current match in $M$.

Can you point to any unstable pairings in this matching?

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- \( h \) prefers \( s \) to its current match in \( M \), and
- \( s \) prefers \( h \) to its current match in \( M \)

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(Beth, MA) are better off together

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Can you point to any unstable pairings in this matching?
**Stable Matching Problem**

**Problem.** Given the preference lists of \( n \) hospitals and \( n \) students, find a **perfect stable** matching, that is, matching \( M \) where:

- every doctor is assigned to a single hospital, and every hospital is assigned to a single doctor, and

- no hospital \( h \) and doctor \( d \) would both prefer to leave their current match to join each other.

**Question.** Does such a matching always exist?

The answer to this does not seem obvious!
A First Attempt

Proceed **greedily** in rounds until matched. In each round:

- Each hospital makes an offer to its top available candidate
- Each doctor *accepts* its top offer (irrevocable contract) and *rejects* any others

Does anything go wrong? Let’s try it!

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A First Attempt

Proceed greedily in rounds until matched.

• (Round 1) MA → Aamir, NH → Aamir, OH → Chris

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What does Amir do? What does Beth do? What does Chris do?
A First Attempt

Proceed greedily in rounds until matched.

- (Round 1) MA → Aamir, NH → Aamir, OH → Chris
- (Round 1) Aamir rejects MA, accepts NH, Chris accepts OH

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A First Attempt

Proceed greedily in rounds until matched.

- (Round 1) MA → Aamir, NH → Aamir, OH → Chris
- (Round 1) Aamir rejects MA, accepts NH, Chris accepts OH
- (Round 2) Only Beth and MA left, and must match

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- (Round 1) MA → Aamir, NH → Aamir, OH → Chris
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Is this a stable matching?

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A First Attempt

Proceed greedily in rounds until matched.

- (Round 1) MA → Aamir, NH → Aamir, OH → Chris
- (Round 1) Aamir rejects MA, accepts NH, Chris accepts OH
- (Round 2) Only Beth and MA left, and must match

Is this a stable matching?

- No! Unstable pair: (MA, Chris). What could have avoided it?

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False Starts are a Problem.

• We want to prove: a perfect stable matching always exists
• One way:
  • Give an algorithm to find a stable matching
  • Prove that it is always successful
  • Constructive method
• Luckily, we now have some insights from our failed attempt, so let’s look at the…

Gale-Shapely Deferred Acceptance Algorithm*
Propose-Reject Algorithm

Initialize each doctor $d$ and hospital $h$ as *Free*

**while** there is a free doctor who hasn’t proposed to every hospital **do**

Choose a free doctor $d$

$h ←$ first hospital on $d$’s list to whom $d$ has not yet proposed

**if** $h$ is *Free then*

$d$ and $h$ are *Matched*

**else if** $h$ prefers $d$ to its current match $d'$ **then**

$d$ and $h$ are *Matched* and $d'$ is *Free*

**else**

$h$ rejects $d$ and remains *Free*

**end if**

**end while**
Observations

(Write these down, we’ll use them later)

**Observation 1.** A doctor proposes at most $n$ times, to $n$ different hospitals.
Propose-Reject Algorithm

Initialize each doctor \( d \) and hospital \( h \) as \textit{Free}

\textbf{while} there is a free doctor who hasn’t proposed to every hospital \textbf{do}

Choose a free doctor \( d \)

\( h \leftarrow \) first hospital on \( d \)'s list to whom \( d \) has not yet proposed

\textbf{if} \( h \) is \textit{Free} \textbf{then}

\( d \) and \( h \) are \textit{Matched}

\textbf{else if} \( h \) prefers \( d \) to its current match \( d' \) \textbf{then}

\( d \) and \( h \) are \textit{Matched} and \( d' \) is \textit{Free}

\textbf{else}

\( h \) rejects \( d \) and remains \textit{Free}

\textbf{end if}

\textbf{end while}
Observations

(Write these down, we’ll use them later)

**Observation 1.** A doctor proposes at most $n$ times, to $n$ different hospitals.

**Observation 2.** Once a hospital is matched, it never becomes unmatched, it only “trades up”.
Propose-Reject Algorithm

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**Observation 1.** A doctor proposes at most $n$ times, to $n$ different hospitals.

**Observation 2.** Once a hospital is matched, it never becomes unmatched, it only “trades up”.

Now let’s make and prove some claims about the algorithm.

(By explicitly stating and labeling our observations, we can refer to them in our proofs!)
Algorithm Analysis

Claim 1. The propose-reject algorithm terminates after at most $n^2$ iterations of the while loop.

Proof. The proof directly analyzes the structure of the algorithm.

1. A doctor proposes during each iteration of the while loop.
Propose-Reject Algorithm

Initialize each doctor $d$ and hospital $h$ as *Free*

while there is a free doctor who hasn’t proposed to every hospital do

Choose a free doctor $d$

$h \leftarrow$ first hospital on $d$’s list to whom $d$ has not yet proposed

if $h$ is *Free* then

$d$ and $h$ are *Matched*

else if $h$ prefers $d$ to its current match $d'$ then

$d$ and $h$ are *Matched* and $d'$ is *Free*

else

$h$ rejects $d$ and remains *Free*

end if

end while
Claim 1. The propose-reject algorithm terminates after at most $n^2$ iterations of the \textbf{while} loop.

\textit{Proof.} The proof directly analyzes the structure of the algorithm.

1. A doctor proposes during each iteration of the \textbf{while} loop.

2. Since there are $n$ doctors and each can propose to at most $n$ different hospitals, the \textbf{while} loop can execute at most $n^2$ times.
Algorithm Analysis

Claim 2. The propose-reject algorithm returns a perfect matching.

Proof. The proof is by contradiction. Suppose the algorithm yields an imperfect matching.

1. Since we do not allow many-to-one relationships, there must be both a doctor $d$ and a hospital $h$ who are unmatched.

2. By Observation 2, $h$ was never proposed to by anyone, which includes $d$.

3. But if $d$ is still free, then, by the while loop condition, $d$ must have proposed to every hospital, including $h$. This is a contradiction.
Algorithm Analysis

Claim 3. The perfect matching yielded by the algorithm is stable.

Proof. The proof is by contradiction. Suppose the algorithm yields an unstable perfect matching.

1. Then there exist two pairs \((d_1, h_1)\) and \((d_2, h_2)\) such that \(d_1\) and \(h_2\) prefer each other to their current assignment.
   In other words, the rankings look something like:
   \[
   d_1 : \ldots, h_2, \ldots, h_1, \ldots \quad \text{and} \quad h_2 : \ldots, d_1, \ldots, d_2, \ldots.
   \]

2. Since \(d_1\) ranks \(h_2\) higher than \(h_1\), \(d_1\) proposed to \(h_2\) sometime before proposing to \(h_1\).

3. But by Observation 2, \(h_2\) only ever trades up, so \(d_2\) must be ranked higher than \(d_1\). This is a contradiction.
What Have We Shown?

So far we have analyzed the algorithm in a couple of ways:

- We proved key properties about its output
  - It yields perfect matchings (Claim 2)
  - It yields stable matchings (Claim 3)
- We showed that the `while` loop executes at most $n^2$ times (Claim 1)
  - **Question**: Does this mean the algorithm is $O(n^2)$?
What Have We Shown?

We’ve specified the algorithm using a powerful and abstract pseudocode.

• Our pseudocode *ignores data representation*

We can reason about correctness, but not efficiency.

• Efficiency comes when we add the data structures!
Representing the Input

**Idea:** Order the doctors arbitrarily from 1 to \( n \). Similarly, arbitrarily order the hospitals from 1 to \( n \). A ranking list for doctors is an \( n \times n \) matrix \( D \), where position \( D(i, j) \) gives the \( j^{th} \) favorite hospital for doctor \( i \). Similarly, construct matrix \( H \) for hospitals.

Aamir (1), Beth (2), Chris (3)
MA (1), NH (2), OH (3)

<table>
<thead>
<tr>
<th></th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
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</thead>
<tbody>
<tr>
<td>Aamir</td>
<td>OH</td>
<td>NH</td>
<td>MA</td>
</tr>
<tr>
<td>Beth</td>
<td>MA</td>
<td>OH</td>
<td>NH</td>
</tr>
<tr>
<td>Chris</td>
<td>MA</td>
<td>NH</td>
<td>OH</td>
</tr>
</tbody>
</table>

Doctor 1 (Aamir) ranks Hospital 3 (OH) first
Doctor 3 (Chris) ranks Hospital 2 (NH) second
Identifying Free Doctors

**Idea**: Use a queue! A doubly-linked list allows enqueuing and dequeuing in $O(1)$ time.

- Each doctor that is free is stored in the queue.
- Matching a doctor means dequeuing them
- Unmatching means putting the doctor back into the queue.
Identifying Next Proposal

For each doctor, we need to know the highest ranked hospital that they have not yet proposed to.

**Idea**: A particular doctor’s preferences are represented by a row in the matrix $D$. A given doctor $i$ will propose in preference order, i.e., from left to right across row $i$.

For each doctor, maintain a counter that is incremented after each proposal. The counter for doctor $i$ is the index into the preference array at row $i$ of $D$. 
Tracking Matches

We need to know which doctor is matched to which hospital (and vice versa). Since matchings are symmetric, we only need to keep track of one direction.

Idea: Keep track of each hospital’s match using an array of length $n$. Call this array $matched$.

$matched(i) = j$ means that hospital $i$ is matched to doctor $j$

$matched(i) = -1$ means that hospital $i$ is unmatched
Tracking Hospital Preferences

We need to know if a hospital $h$ prefers its current partner to the doctor who just proposed to it.

**Idea**: Create what is called an inverted index of the $H$ matrix (hospital preference matrix), which we will call $R$ ($R$ for ranks). For a given hospital, $R$ doesn’t store it’s preference list; instead, $R$ stores the rank ($1$ to $n$) of each doctor. So to compare a hospital $h$’s ranking of two doctors, $i$ and $j$, we can check $R(h, i)$ and compare it to $R(h, j)$.

We can build the inverted index in $O(n^2)$ time by consulting $H$ (a one-time setup cost), and with it, we compare two doctors rankings in $O(1)$ time.
Inverted Index Example

• Let's use our running example where we’ve numbered our 3 hospitals and 3 doctors as follows:
  Doctors 1: Aamir, 2: Beth, 3: Chris  
  Hospitals 1: MA, 2: NH, and 3: OH

• In our hospital preference table (left), each row specifies a hospital's preferences for doctors in descending order. So in a given hospital row, the first column is the hospital's first choice, the second column second...

• In our inverted index (right), each row specifies a hospital’s ranks for doctors, indexed using the doctors’ numbers. So in a given hospital row, the first column is the ranking of the first doctor, the second column is the ranking of the second doctor...
  • \(R(i, j)\) stores the hospital \(i\)'s ranking (ranging from 1…\(n\)) for doctor \(j\)

<table>
<thead>
<tr>
<th>Hospital Preferences (visual)</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
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</thead>
<tbody>
<tr>
<td>MA</td>
<td>Aamir</td>
<td>Chris</td>
<td>Beth</td>
</tr>
<tr>
<td>NH</td>
<td>Aamir</td>
<td>Beth</td>
<td>Chris</td>
</tr>
<tr>
<td>OH</td>
<td>Chris</td>
<td>Beth</td>
<td>Aamir</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inverted Index (R)</th>
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<th>3</th>
<th>2</th>
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<tr>
<td>1</td>
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<td>3</td>
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</table>

\(R(1,3)\): Hospital 1 (MA) ranks Doctor 3 (Chris) second

\(R(3,2)\): Hospital 3 (OH) ranks Doctor 2 (Beth) second
Inverted Index Example

- We can query the inverted index in $O(1)$ to check if a hospital prefers one doctor to another.

- Suppose we wanted to check whether NH prefers Chris or Aamir:
  - NH is hospital 2, Chris is doctor 3, and Aamir is doctor 1.
  - $R(2,3)$ stores NH's ranking for Chris, and $R(2,1)$ stores NH's ranking for Aamir:
    - $R(2,3) = 3$, while $R(2,1) = 1$, so Aamir is ranked higher!

**Inverted Index** $R$

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<thead>
<tr>
<th></th>
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<td>1</td>
</tr>
</tbody>
</table>

**Doctors** 1: Aamir, 2: Beth, 3: Chris, and

**Hospitals** 1: MA, 2: NH, and 3: OH.
Propose-Reject Algorithm

Initialize each doctor \( d \) and hospital \( h \) as Free

**while** there is a free doctor who hasn’t proposed to every hospital **do**

- Choose a free doctor \( d \)
- \( h \leftarrow \) first hospital on \( d \)'s list to whom \( d \) has not yet proposed

**if** \( h \) is Free **then**

- \( d \) and \( h \) are Matched

**else if** \( h \) prefers \( d \) to its current match \( d' \) **then**

- \( d \) and \( h \) are Matched and \( d' \) is Free

**else**

- \( h \) rejects \( d \) and remains Free

**end if**

**end while**