# Randomized Quicksort

# Randomized Algorithms & Data Structures

- *Monte-Carlo algorithms*
    - Find the correct answer most of the time
    - Can usually amplify probability of success with repetitions
    - Example, Karger's min cut (in textbook)
- *Las-Vegas algorithms*
    - Always find the correct answer, e.g. RandQuick sort (today!)
    - But the worst-case running-time guarantees are not strong (they hold in expectation or with high probability, but their goodness depends on randomness)
- *Randomized data structures*: hashing, search trees, filters, etc.


# Randomized Algorithm I

## Randomized Selection

# Randomized Selection

**Problem.** Find the $k^{\text{th}}$ smallest/largest element in an unsorted array

- Recall our selection algorithm from back in our divide and conquer unit (lecture 15):

**Select** $(A, k)$**:**

If $|A| = 1$: return $A[1]$

Else:

Choose a pivot $p \leftarrow A[1,\ldots,n]$; let $r$ be the rank of $p$

$r, A_{<p}, A_{>p} \leftarrow$ Partition($(A, p)$)

If $k == r$: return $p$

Else if $k < r$: Select $(A_{<p}, k)$

Else: Select $(A_{>p}, k - r)$

# Selection with a Good Pivot

- Recall: pivot is "good" if it reduced the array size by at least a constant
  - Gives a recurrence $T(n) \leq T(\alpha n) + O(n)$ for some constant $\alpha < 1$
  - Expands to a decreasing geometric series $T(n) = O(n)$
- In the deterministic algorithm, how did we find a good pivot?
  - Split array into groups of $5$
  - And computed the median of group medians
  - The pivot guaranteed that $n \to 7n/10$
- **Here is a silly idea:** What if we pick the pivot uniformly at random?
  - Seems like the pivot is "usually" around the midpoint
  - What is the expected running time?

# Randomized Selection

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- Recall our selection algorithm

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Choose a pivot $p \leftarrow A[1,\ldots,n]$ uniformly at random; let $r$ be the rank of $p$

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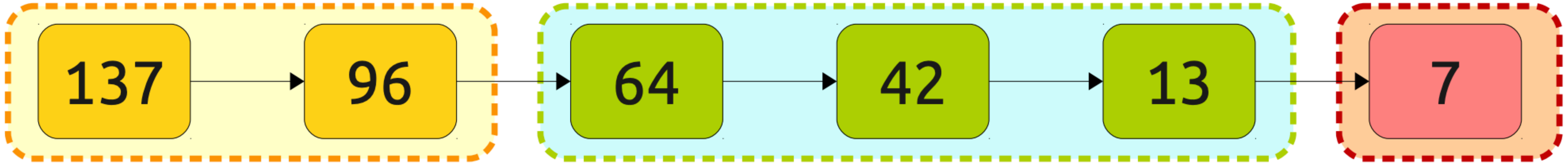
# Analyzing Randomized Selection

- Normally, we'd write a recurrence relation for a recursive function
- A bit complicated now—input sizes of later recursive calls depend on the random choices of pivots in earlier calls
- We will use a different accounting trick for running time
- Randomized selection makes at most one recursive call each time:
  - Group multiple recursive call in "phases"
  - Sum of work done by all calls is equal to the sum of the work done in all the phases



# Analyzing in Phases

- **Idea**: let a "phase" of the algorithm be the time it takes for the array size to drop by a constant factor (say $n \to (3/4) \cdot n$)
- If array shrinks by a constant factor in each phase and linear work is done in each phase, what would be the running time?
- $T(n) = c(n + 3n/4 + (3/4)^2 n + \ldots + 1) = O(n)$
- If we want a $1/4$th, $3/4$th split, what range should our pivot be in?
  - Middle half of the array (if $n$ size array, then pivot in $[n/4, 3n/4]$)
  - What is the probability of picking such a pivot?
    - $1/2$
  - Phase ends as soon as we pick a pivot in the middle half
    - Expected # of recursive calls until phase ends? $2$

# Expected Running Time

- Let the algorithm be in phase $j$ when the size of the array is
    - At least $n\left(\frac{3}{4}\right)^{j+1}$ but not greater that $n\left(\frac{3}{4}\right)^{j}$
- Expected number of iterations within a phase: $2$
- Let $X_j$ be the expected number of steps spent in phase $j$
- $X = X_0 + X_1 + X_2 \ldots$ be the total number of steps taken by the algorithm
- $\mathrm{E}(X_j) = \mathrm{E}(\#$ recursive calls until $j$th phase ends $\cdot$ # steps in phase $j)$
- $\mathrm{E}(X_j) \leq cn(3/4)^j \cdot \mathrm{E}(\#$ recursive calls until $j$th phase ends$) = 2cn(3/4)^j$

# Expected Running Time

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- $X = X_0 + X_1 + X_2 \ldots$ be the total number of steps taken by the algorithm
- $\mathrm{E}(X_j) =$ $\mathrm{E}($# of iterations until $j$th phase ends · # steps in phase $j)$
- $\mathrm{E}(X_j) \leq n(3/4)^j \cdot \mathrm{E}($# iterations until $j$th phase ends$)$ $= 2cn(3/4)^j$
- Now we can apply linearity of expectation:
- $$E[X] = \sum_j E[X_j] \leq \sum_j 2cn \left(\frac{3}{4}\right)^j = 2cn \sum_j \left(\frac{3}{4}\right)^j$$

$$= \Theta(n)$$

# Pivot Selection

- Deterministic and random both take $O(n)$ time
  - What's the advantage of the deterministic algorithm?
  - Worst-case guarantee—the random algorithm could be very slow sometimes
  - What's the advantage of the random algorithm?
  - Much much simpler and better constants hidden in $O()$
- Which should you use?
  - Pretty much always random
  - Question to ask yourself:
    - how often is the randomized algorithm going to be much worse than $O(n)$?

# Acknowledgments


- Some of the material in these slides are taken from
  - Shikha Singh
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)