Wrapping Up Our NP Hardness Reductions

Grouping Problems

- The textbook does a good job grouping problems
    - Packing problems (Independent set)
    - Covering problems (Vertex cover, set cover)
    - 8.5: Sequencing Problems (Hamiltonian cycle/path, traveling salesmen)
    - 8.6 Partitioning Problems (3-dimensional matching)
    - 8.7 Graph Coloring (3-coloring)
    - 8.8 Numerical Problems (Subset sum, knapsack)
- If your problem seems to fit into one of these categories, it is a reasonable strategy to pick a problem from the same category as a reduction candidate

Today’s Plan

- We are finishing our in-class exploration of reductions
    - We’ll show that subset sum is NP-complete
- *Didn’t we say that subset sum had an O(nW) dynamic programming algorithm, just like knapsack?*
    - Yes, but O(nW) is not polynomial with respect to the input size (representation); W is a value!
- Steps to show subset sum is NP-complete?
    - Show subset sum is in NP
    - Reduce a known NP-complete problem to it (in poly-time)
    - Prove yes instances map to yes instances (both ways)

SUBSET-SUM is NP Complete:

Vertex-Cover $\leq_p$ SUBSET-SUM

Subset Sum Problem

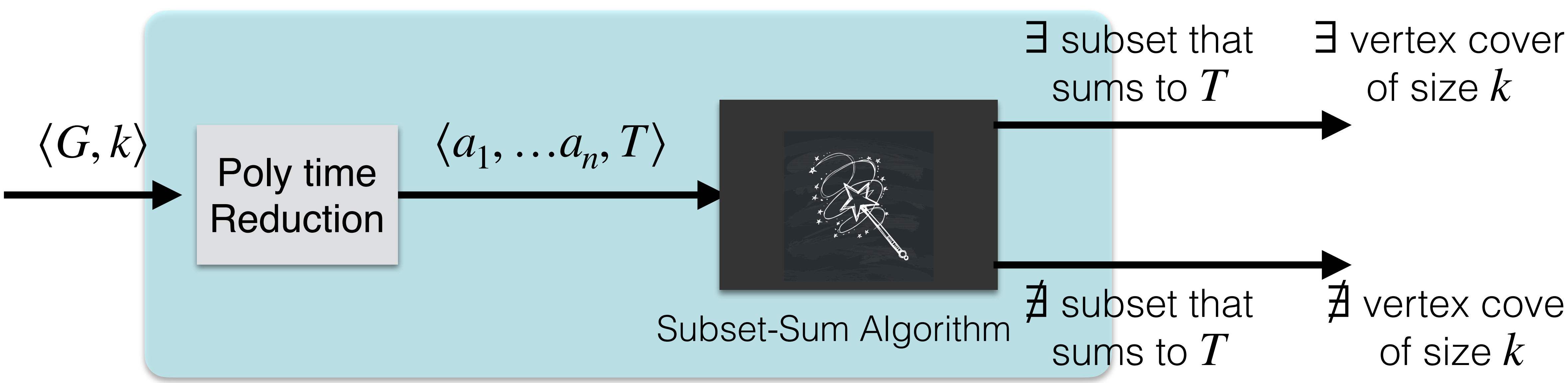
- **SUBSET-SUM.**
  Given n positive integers $a_1, \ldots, a_n$ and a target integer T, is there a subset of numbers that adds up to exactly T
- Step 1: show that **SUBSET-SUM** $\in$ NP
  - Certificate: a set of numbers
  - Poly-time verifier: checks if the set is a subset of the input integers, and if so, that the set sums exactly to T
- Step 2: prove that **SUBSET-SUM** is **NP hard** by reducing a known NP-complete problem to it
  - We'll reduce from vertex cover

Map the Problems

Vertex Cover	What is a possible solution?	**Subset Sum**
A selection of vertices to be in vertex cover C		A subset of integers $S \subseteq \{a_1, \ldots, a_n\}$
	What is the requirement?	
C must contain at most k vertices		numbers in S must sum to T
	What are the restrictions?	
If $(u, v) \in E$, then either u or v must be in S		S must be a subset of input integers

Vertex Cover to Subset Sum

- **Theorem.** VERTEX-COVER $\leq_p$ SUBSET-SUM
- **Proof**. Given a graph G with n vertices and m edges and a number k, we construct a set of numbers $a_1, \ldots, a_t$ and a target sum T such that G has a vertex cover of size k iff there is a subset of numbers that sum to T



Algorithm for Vertex Cover

Vertex Cover to Subset Sum

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  - In our reduction, we'll use two types of **gadgets**
    - when reducing a problem X to a problem Y, a gadget is a small (partial) instance of problem Y that is used to "simulate" a feature of problem X
    - Gadget 1: integers that represent vertices of G, and
    - Gadget 2: integers that represent edges of G

Vertex Cover to Subset Sum

- **Theorem.** VERTEX-COVER $\leq_p$ SUBSET-SUM
- **Reduction (idea)**
  - We'll create a set of $n + m$ integers, where we have one integer for every vertex, and one integer for every edge ($A = \{a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}\}$)
  - **Goals** when creating our vertex/edge gadgets:
    - Must force the selection of k vertex integers: need to ensure that no other set of input integers can sum to T
    - Must force an edge covering: for every edge (u, v), we need to ensure that our subset can't sum to T unless either u or v is picked

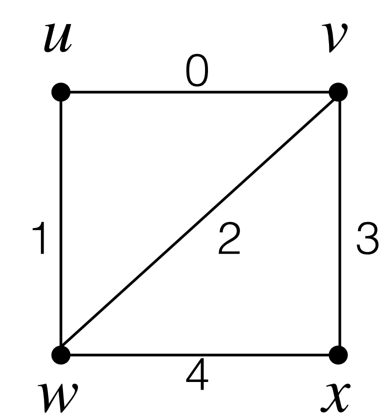
Vertex Cover to Subset Sum

- **Theorem.** VERTEX-COVER $\leq_p$ SUBSET-SUM
- **Reduction**.
  First, arbitrarily number the edges from 0 to $m-1$. Then, create set of $n+m$ integers and a target value T as follows:
  - Vertex integer a_v : m^{th} (most significant) bit is 1 and for $i<m$, the i^{th} bit is 1 if i^{th} edge is incident to vertex v
  - Edge integer b_{uv} : m^{th} digit is 0 and for $i<m$, the i^{th} bit is 1 if this integer represents an edge $i=(u,v)$
  - Target value $T = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$

Note: Each integer is a $m+1$-bit number in base four*

Vertex Cover to Subset Sum

- Example: consider the graph $G = (V, E)$ where $V = \{u, v, w, x\}$ and $E = \{(u,v),\ (u,w),\ (v,w),\ (v,x),\ (w,x)\}$



	5th	4th : (wx)	3rd : (vx)	2nd : (vw)	1st : (uw)	0th: (uv)
a_u	1	0	0	0	1	1
a_v	1	0	1	1	0	1
a_w	1	1	0	1	1	0
a_x	1	1	1	0	0	0
b_{uv}	0	0	0	0	0	1
b_{uw}	0	0	0	0	1	0
b_{vw}	0	0	0	1	0	0
b_{vx}	0	0	1	0	0	0
b_{wx}	0	1	0	0	0	0

$$
\begin{aligned}
a_u &:= 111000_4 = 1344 \\
a_v &:= 110110_4 = 1300 \\
a_w &:= 101101_4 = 1105 \\
a_x &:= 100011_4 = 1029
\end{aligned}
$$

$$
\begin{aligned}
b_{uv} &:= 010000_4 = 256 \\
b_{uw} &:= 001000_4 = 64 \\
b_{vw} &:= 000100_4 = 16 \\
b_{vx} &:= 000010_4 = 4 \\
b_{wx} &:= 000001_4 = 1
\end{aligned}
$$

- If $k = 2$ then $T = 222222_4 = 2730$

Correctness

- **Claim.** G has a vertex cover of size k if and only there is a subset X of corresponding integers that sums to value T
- ($\Rightarrow$) Let C be a vertex cover of size k in G, define X as
  $X := \{a_v \mid v \in C\} \cup \{b_i \mid \text{edge } i \text{ has exactly one endpoint in } C\}$

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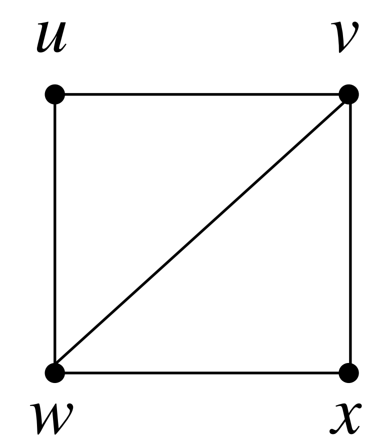
a_x	1	1	1	0	0	0
b_{uv}	0	0	0	0	0	1
b_{uw}	0	0	0	0	1	0
b_{vw}	0	0	0	1	0	0
b_{vx}	0	0	1	0	0	0
b_{wx}	0	1	0	0	0	0

$C = \{v, w\}$



$T = 222222_4 = 2730$

$$T = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$$

Correctness

- **Claim.** G has a vertex cover of size k if and only there is a subset X of corresponding integers that sums to value T
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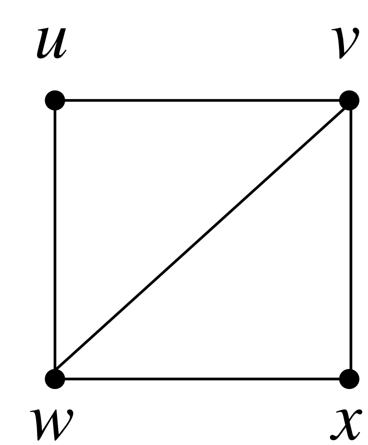
  $X := \{a_v \mid v \in C\} \cup \{b_i \mid$ edge i has exactly one endpoint in $C\}$

	5^{th}	4^{th} : (wx)	3^{rd} : (vx)	2^{nd} : (vw)	1^{st} : (uw)	0^{th} : (uv)
[redacted]						
a_v	1	0	1	1	0	1
a_w	1	1	0	1	1	0
[redacted]						
b_{uv}	0	0	0	0	0	1
b_{uw}	0	0	0	0	1	0
[redacted]						
b_{vx}	0	0	1	0	0	0
b_{wx}	0	1	0	0	0	0

$$C = \{v, w\}$$



$$T = 222222_4 = 2730$$

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Correctness

- **Claim.** G has a vertex cover of size k if and only there is a subset X of corresponding integers that sums to value T
- $(\Rightarrow)$ Let C be a vertex cover of size k in G, define X as
  $X := \{a_v \mid v \in C\} \cup \{b_i \mid$ edge i has exactly one endpoint in $C\}$
- Sum of the most significant bits of X is k
  - Only vertex gadgets have MSB set, select k vertex integers
- All other bits must sum to 2, why?
- Consider column for edge (u, v):
  - Either both endpoints are in C, then we get two 1's from a_v and a_u and none from b_{uv}
  - Exactly one endpoint is in C: get 1 bit from b_{uv} and 1 bit from a_u or a_v
- Thus the elements of X sum to exactly T

Vertex Cover to Subset Sum

- **Claim.** G has a vertex cover of size k if and only there is a subset X of corresponding integers that sums to value T
- $(\Leftarrow)$ Let X be the subset of numbers that sum to T
- That is, there is $V' \subseteq V, E' \subseteq E$ s.t.

$$X := \sum_{v \in V'} a_v + \sum_{i \in E'} b_i = T = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$$

- These numbers are base 4 and there are no carries
- Each b_i only contributes 1 to the i^{th} digit, which is 2 in our target T
- Thus, for each edge i, at least one of its endpoints must be in V'
  - V' is a vertex cover
- Size of V' is k: only vertex-numbers have a 1 in the m^{th} position

Subset Sum: Final Thoughts

- Polynomial time reduction?
  - $O(nm)$ since we check vertex/edge incidence for each vertex/edge when creating $n + m$ numbers
- Does a $O(nT)$ subset-sum algorithm mean vertex cover can be solved in polynomial time?
  - No! $T \approx 4^m$
- NP hard problems that have pseudo-polynomial algorithms are called *weakly NP hard*

Steps to Prove X is NP Complete

- Step 1. Show X is in **NP**
- Step 2. Pick a known NP hard problem Y from class
- Step 3. Show that $Y \leq_p X$
    - Show both sides of reduction are correct: if and only if directions
    - State that reduction runs in polynomial time in input size of problem Y

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)