NP Hardness Reductions

Reminders/Check-in

- HW Clarifications
  - We ask you to give a polynomial-time algorithm, so want to justify that your algorithm is polynomial time
    - Rudimentary analysis is OK! But remember that cost of algorithm is cost of reduction + cost of solving/interpreting flow
  - Ford Fulkerson: O(nmC) or O(mC)?
    - Textbook uses different definition of C than we did in our discussion…
- Probability review
  - Readings accessible from on-campus only (or using proxy)

Big Picture

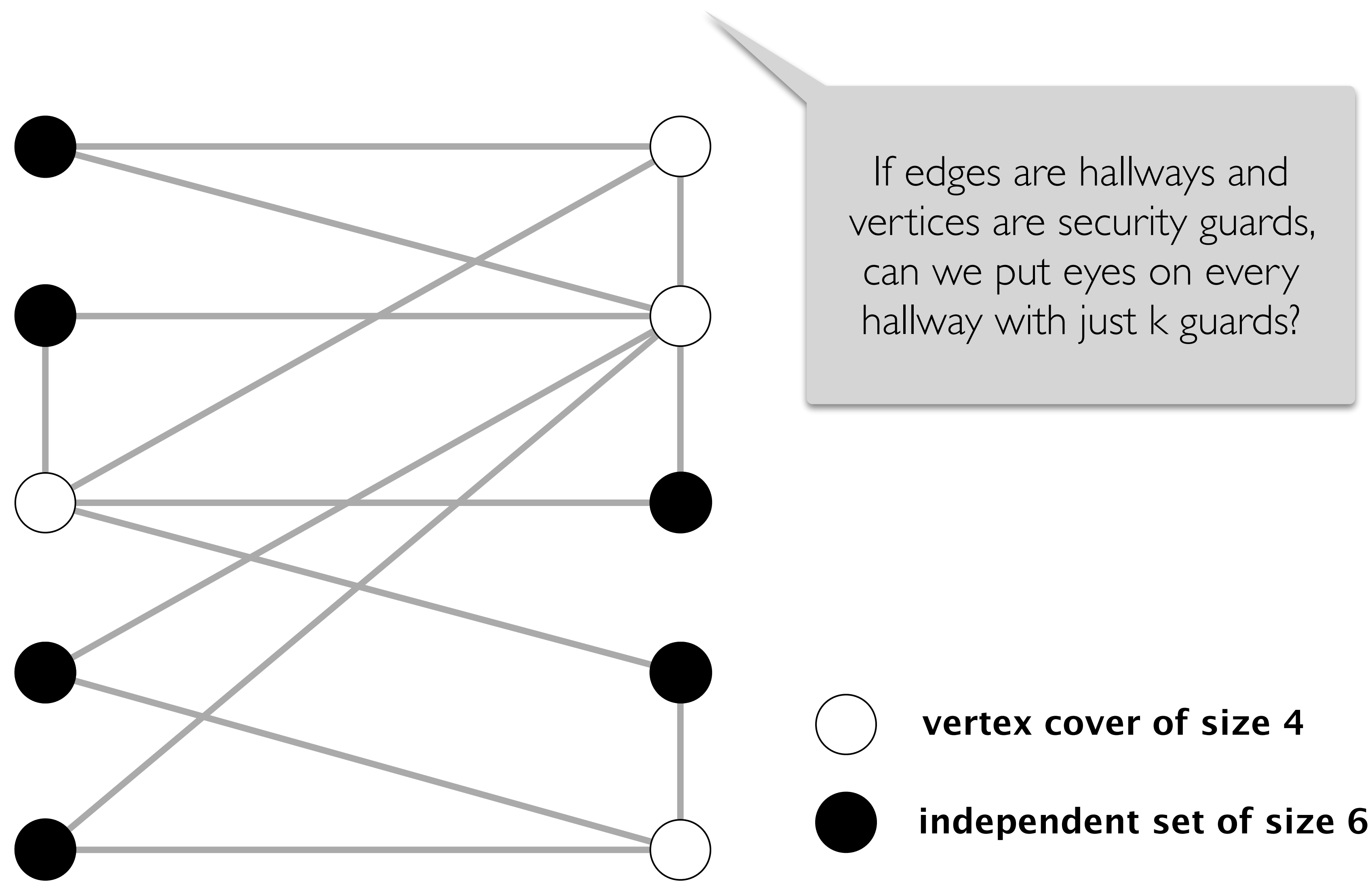
- “Does P = NP?” is an important question in CS
  - Knowing the answer would be nice, but the debate around the question informs our thinking about “hard” problems
- So why are we covering it? What should your takeaways be?
  - Be able to give an operation definition of and describe the P and the NP-complete problem classes
  - Be able to complete and prove a problem reduction beyond the examples we cover together (i.e., apply the reduction framework)
  - Be familiar with a handful of the “classic” NP-hard problems
    - If you hear “Vertex Cover” at a party…

VERTEX-COVER $\leq_p$ SET-COVER

Vertex-Cover

Given a graph $G = (V, E)$, a vertex cover is a subset of vertices $T \subseteq V$ such that for every edge $e = (u, v) \in E$, either $u \in T$ or $v \in T$.

- **VERTEX-COVER <u>decision</u> Problem.** Given a graph $G = (V, E)$ and an integer k, does G have a vertex cover of size at most k?



Set Cover

Set-Cover. Given a set U of elements, a collection $\mathcal{S}$ of subsets of U and an integer k, is there some collection of **at most** k subsets $S_1, \ldots, S_k$ whose union covers U, that is, $U \subseteq \cup_{i=1}^{k} S_i$

$$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$$

$$S_a = \{ 3, 7 \} \qquad S_b = \{ 2, 4 \}$$

$$S_c = \{ 3, 4, 5, 6 \} \qquad S_d = \{ 5 \}$$

$$S_e = \{ 1 \} \qquad S_f = \{ 1, 2, 6, 7 \}$$

$$k = 2$$

a set cover instance

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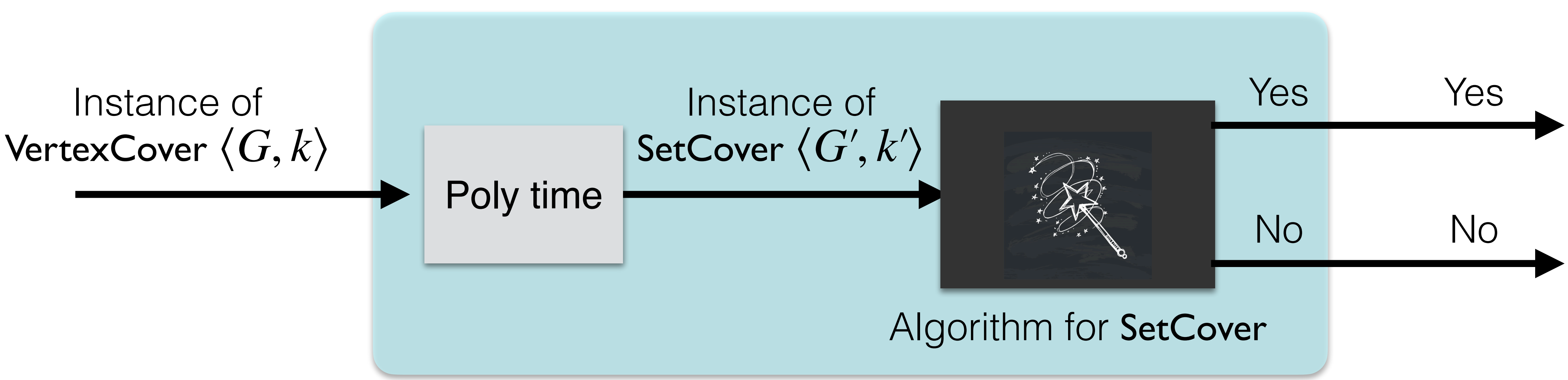
a set cover instance

Vertex Cover $\leq_p$ Set Cover

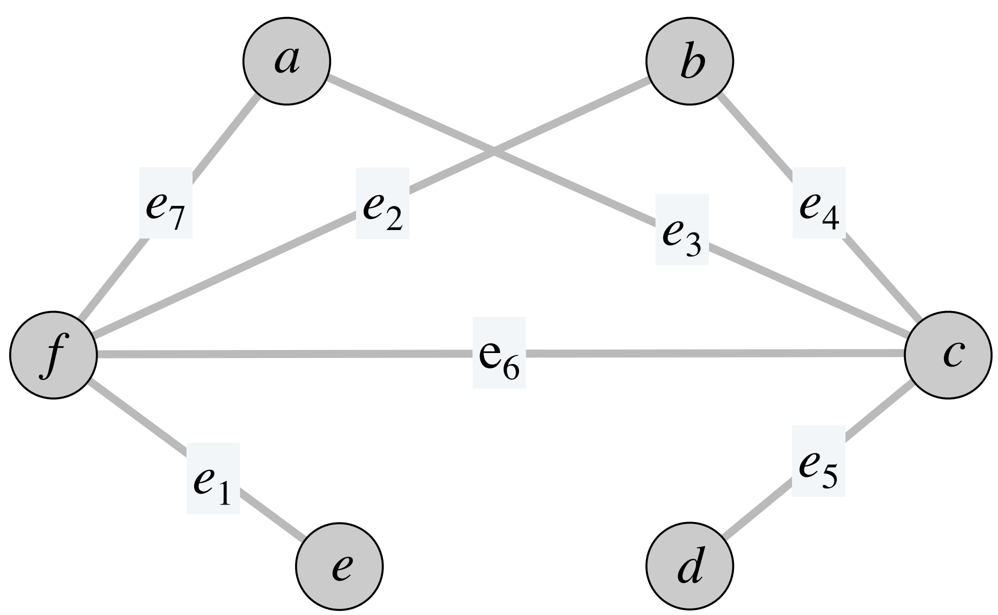
- **Theorem.** VERTEX-COVER $\leq_p$ SET-COVER
- **Proof.** Given instance $\langle G, k \rangle$ of vertex cover, construct an instance $\langle U, \mathcal{S}, k' \rangle$ of set cover problem such that
- G has a vertex cover of size at most k if and only if $\langle U, \mathcal{S}, k' \rangle$ has a set cover of size at most k.



Algorithm for VertexCover

Vertex Cover $\leq_p$ Set Cover

- **Theorem.** VERTEX-COVER $\leq_p$ SET-COVER
- **Proof.** Given instance $\langle G, k \rangle$ of vertex cover, construct an instance $\langle U, \mathcal{S}, k \rangle$ of set cover problem that has a set cover of size k iff G has a vertex cover of size k.
- **Reduction.** $U = E$. $\mathcal{S}$: for each node $v \in V$, let $S_v = \{e \in E \mid e \text{ incident to } v\}$



vertex cover instance
(k = 2)

$U = \{ e_1, e_2, \ldots, e_7 \}$

$S_a = \{ e_3, e_7 \}$ $\quad$ $S_b = \{ e_2, e_4 \}$

$S_c = \{ e_3, e_4, e_5, e_6 \}$ $\quad$ $S_d = \{ e_5 \}$

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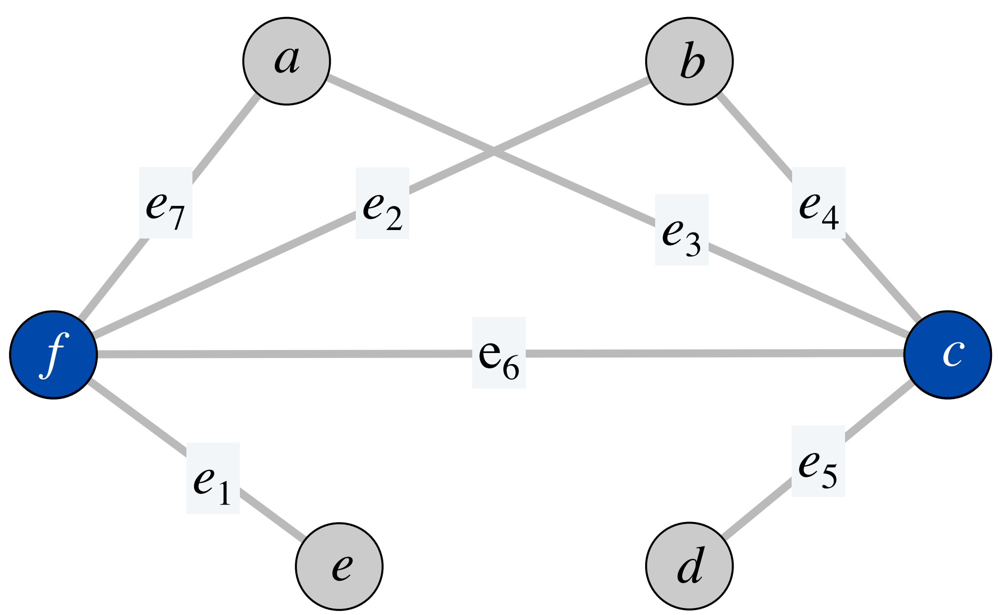
set cover instance
(k = 2)

Correctness

- **Claim.** $(\Rightarrow)$ If G has a vertex cover of size at most k, then U can be covered using at most k subsets.
- **Proof.** Let $X \subseteq V$ be a vertex cover in G
  - Then, $Y = \{S_v \mid v \in X\}$ is a set cover of U of the same size



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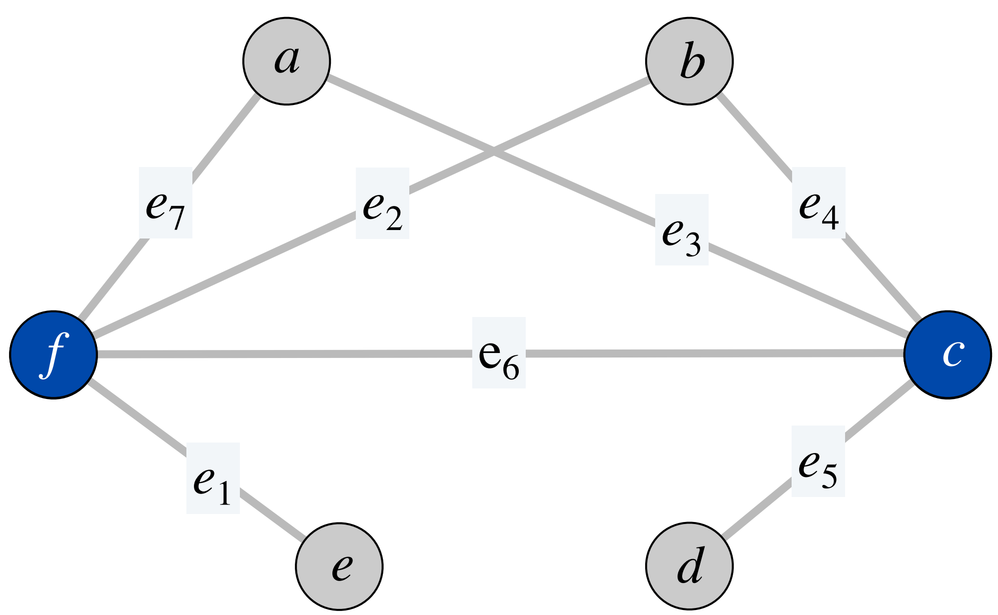
set cover instance
(k = 2)

Correctness

- **Claim.** $(\Leftarrow)$ If U can be covered using at most k subsets then G has a vertex cover of size at most k.
- **Proof.** Let $Y \subseteq \mathcal{S}$ be a set cover of size k
  - Then, $X = \{v \mid S_v \in Y\}$ is a vertex cover of size k



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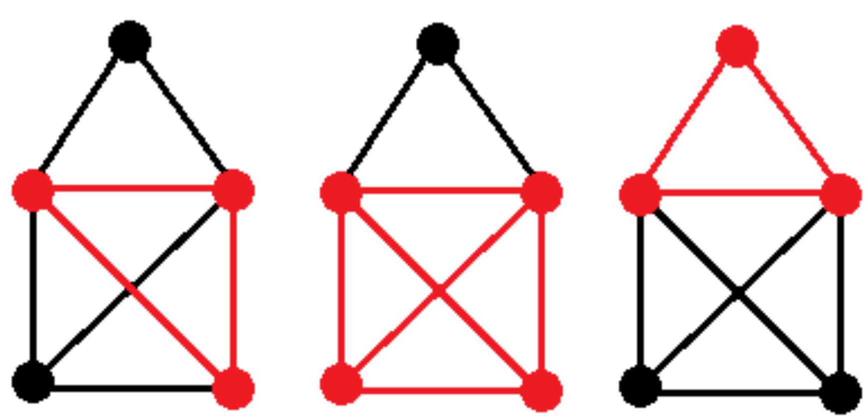
set cover instance
(k = 2)

Class Exercise

IND-SET $\leq_p$ Clique

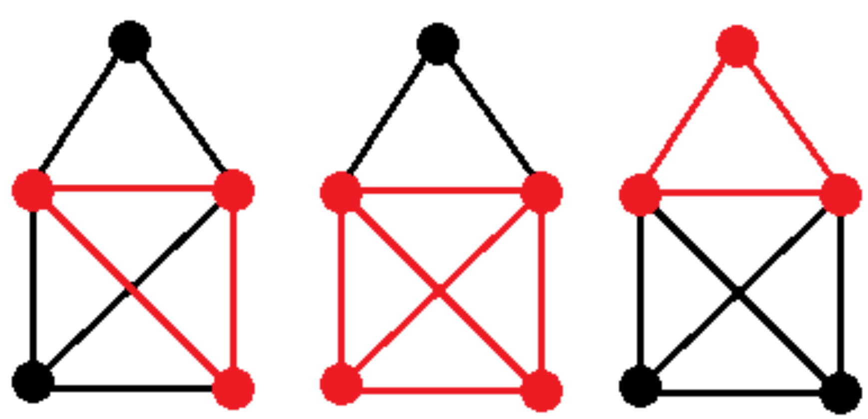
Clique

- A **clique** in an undirected graph is a subset of nodes such that every two nodes are connected by an edge. A k-clique is a clique that contains k nodes.
- **CLIQUE.** Given a graph G and a number k, does G contain a k -clique?



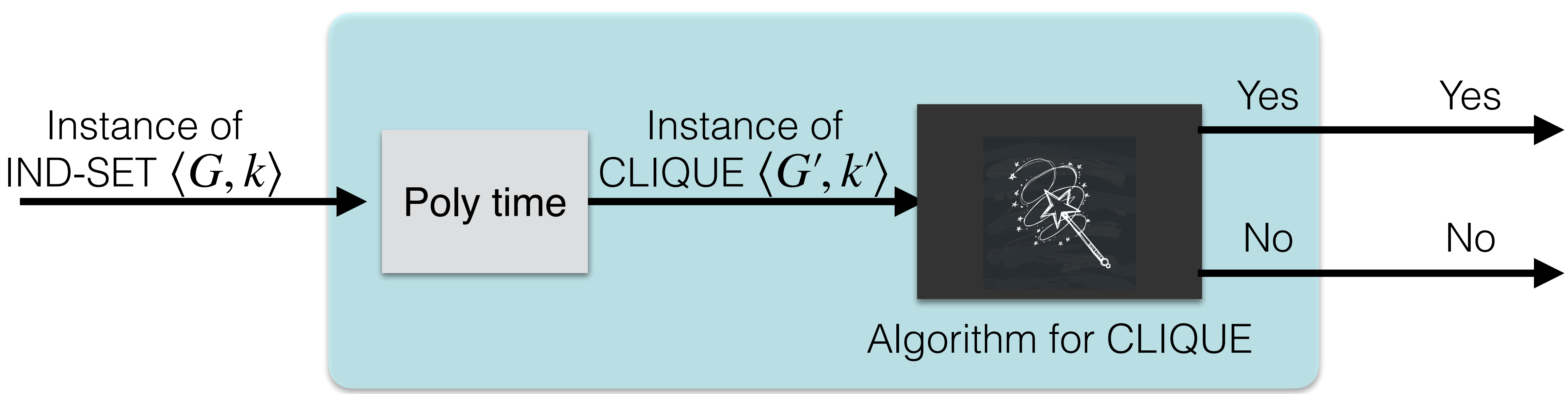
Clique

- A **clique** in an undirected graph is a subset of nodes such that every two nodes are connected by an edge. A k-clique is a clique that contains k nodes.
- **CLIQUE.** Given a graph G and a number k, does G contain a k -clique?
- **CLIQUE** $\in$ NP
    - Certificate: a subset of vertices
    - Poly-time verifier: check is each pair of vertices have an edge between them and if size of subset is k



IND-SET to CLIQUE

- **Theorem.** **IND-SET $\leq_p$ CLIQUE.**
- **In class exercise.** Reduce IND-SET to Clique. Given instance $\langle G, k \rangle$ of independent set, construct an instance $\langle G', k' \rangle$ of clique such that
  - G has independent set of size k iff G' has clique of size k'.

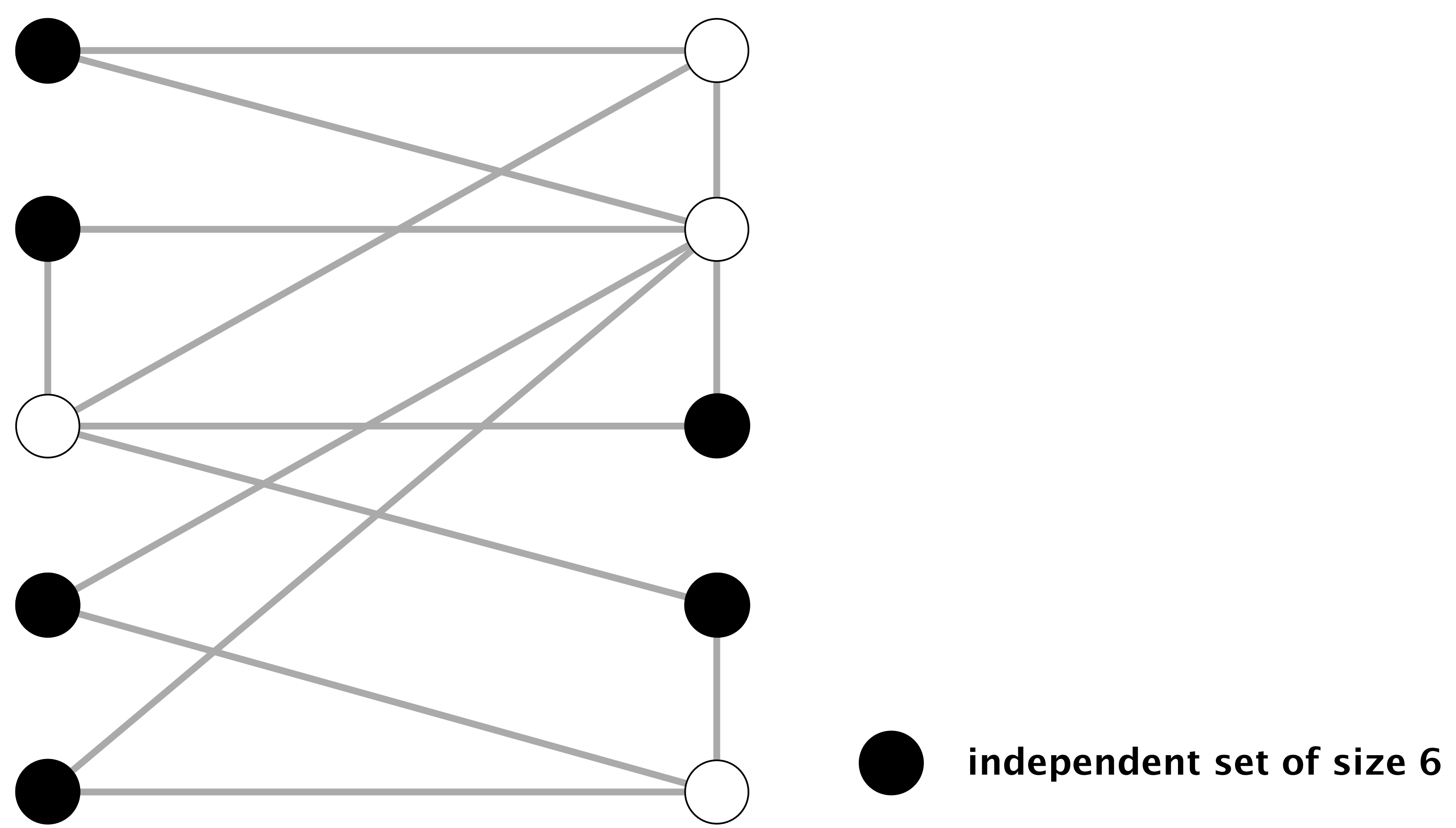


Algorithm for IND-SET

Recall: IND-SET

Given a graph $G = (V, E)$, an independent set is a subset of vertices $S \subseteq V$ such that no two of them are adjacent, that is, for any $x, y \in S$, $(x, y) \notin E$

- **IND-SET <u>decision</u> Problem.** Given a graph $G = (V, E)$ and an integer k, does G have an independent set of size at least k?



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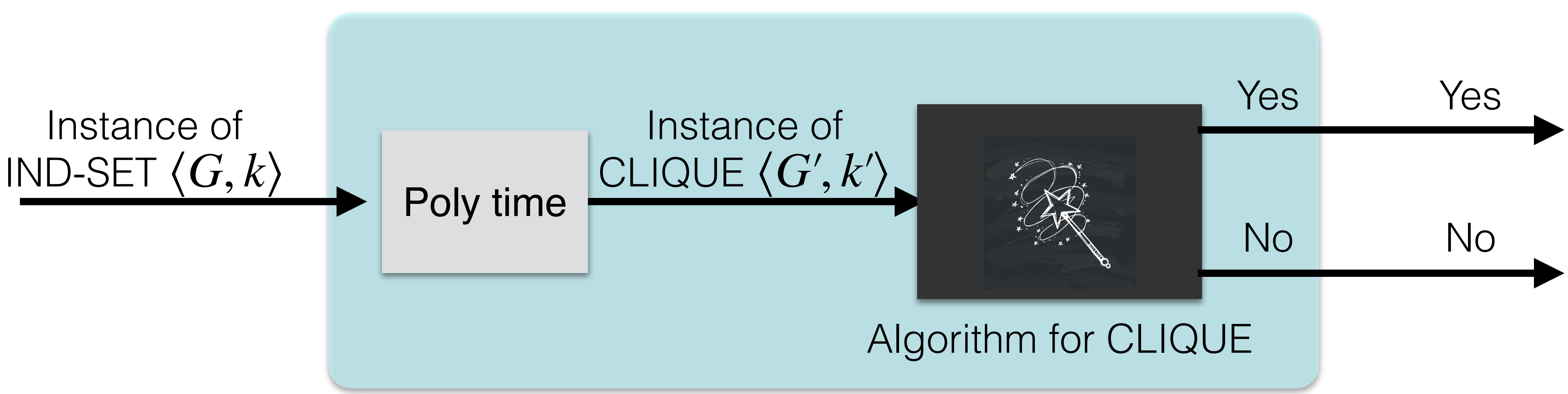


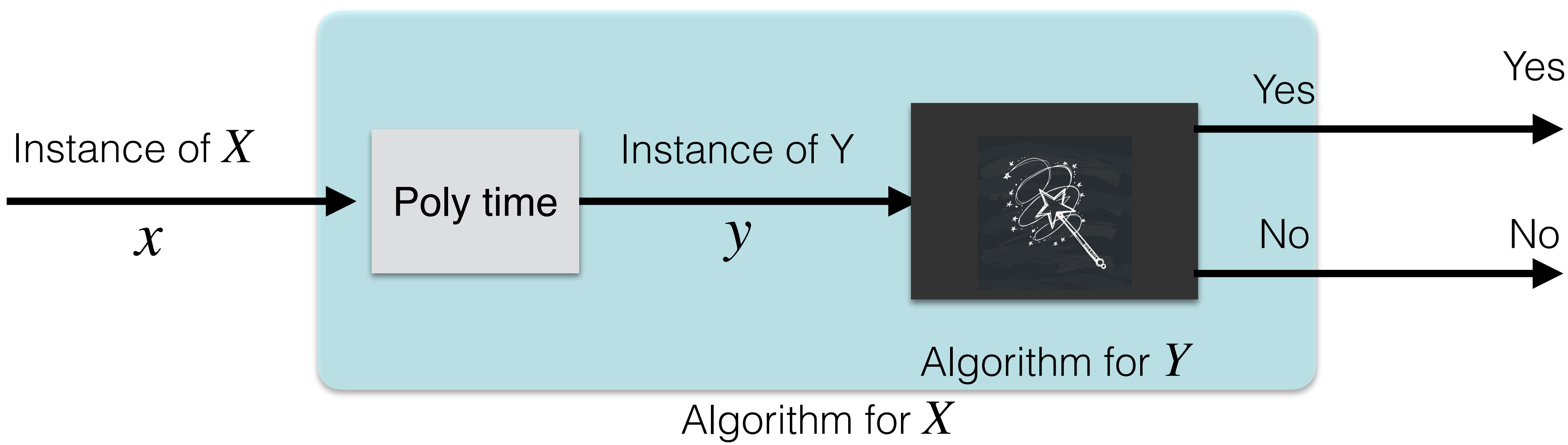
Algorithm for IND-SET

IND-SET to CLIQUE

- **Theorem.** IND-SET $\leq_p$ CLIQUE.
- Proof. Given instance $\langle G, k \rangle$ of independent set, we construct an instance $\langle G', k' \rangle$ of clique such that G has independent set of size k iff G' has clique of size k'
- **Reduction**.
  - Let $G' = (V, \overline{E})$, where $e = (u, v) \in \overline{E}$ iff $e \notin E$ and $k' = k$
  - $(\Rightarrow)$ G has an independent set S of size k, then S is a clique in G'
  - $(\Leftarrow)$ G' has a clique Q of size k, then Q is an independent set in G

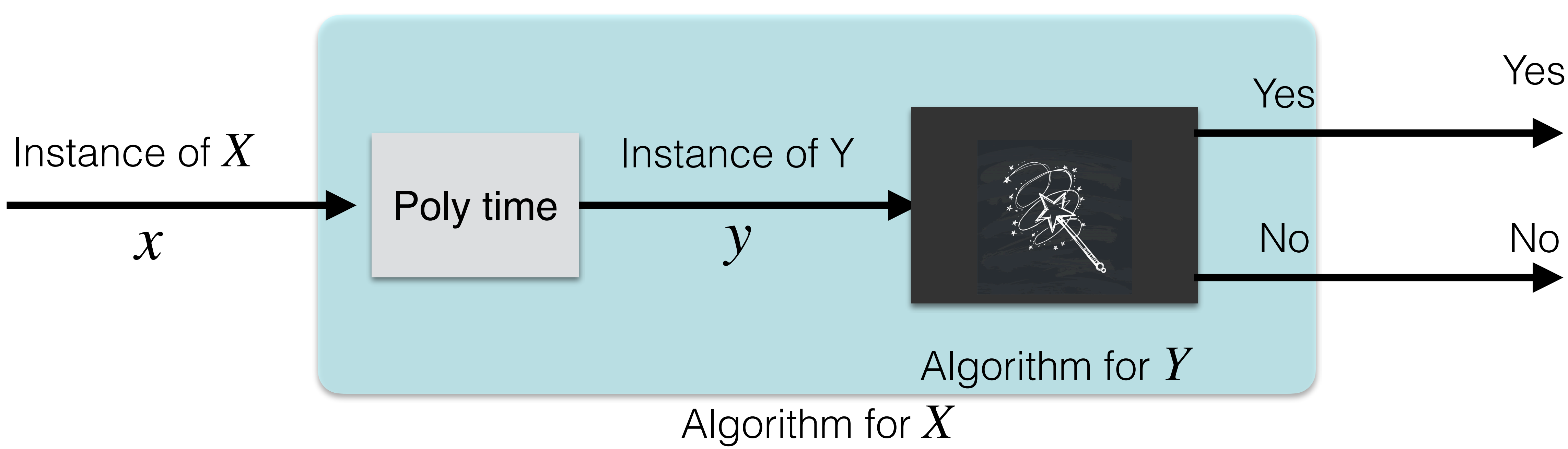
Reductions: General Pattern

- Describe a polynomial-time algorithm to transform an arbitrary instance x of Problem X into a special instance y of Problem Y
- Prove that:
  - If x is a "yes" instance of X, then y is a "yes" instance of Y
  - If y is a "yes" instance of Y, then x is a "yes" instance of X



Reductions: General Pattern

- Describe a polynomial-time algorithm to transform an arbitrary instance x of Problem X into a special instance y of Problem Y
- Notice that correctness of reductions are not symmetric:
    - the "if" proof needs to handle arbitrary instances of X
    - the "only if" needs to handle the special instance of Y



IND-SET is NP Complete:

3SAT $\leq_p$ IND-SET

Problem Definition: 3-SAT

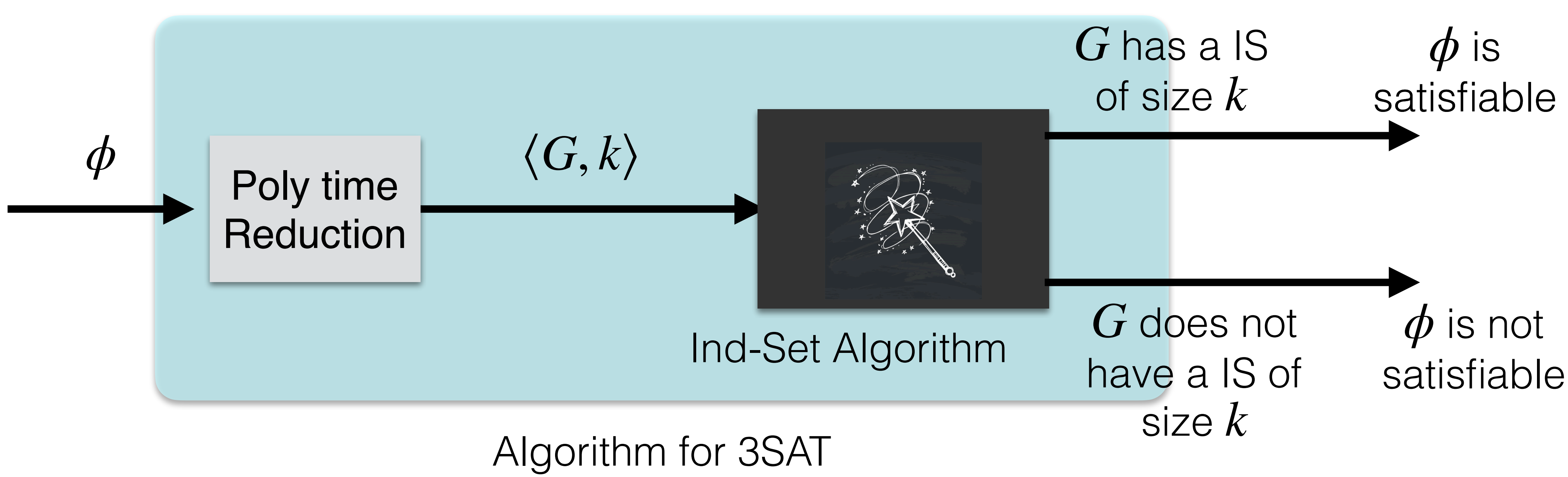
- **Literal**. A Boolean variable or its negation x_i or $\overline{x_i}$
- **Clause**. A disjunction of literals $C_j = x_1 \vee \overline{x_2} \vee x_3$
- **Conjunctive normal form (CNF).** A boolean formula ϕ that is a conjunction of clauses $\Phi = C_1 \wedge C_2 \wedge C_3$
- **SAT**. Given a CNF formula Φ, does it have a satisfying truth assignment?
- **3SAT.** A SAT formula where each clause contains exactly 3 literals (corresponding to different variables)
- $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$
- **SAT**, **3SAT** are both NP complete
- We will use 3SAT to prove other problems are NP hard

IND-SET: NP Complete

- To show Independent set is NP complete
  - Show it is in NP (we've already done this)
  - Reduce a known NP complete problem to it
    - We will use 3-SAT
- Looking ahead: once we have shown 3-SAT $\leq_p$ IND-SET
  - Since **IND-SET** $\leq_p$ **Vertex Cover**
  - And **Vertex Cover** $\leq_p$ **Set Cover**
  - We can conclude they are also NP hard
  - As they are both in NP, they are also NP complete!

IND-SET: NP hard

- **Theorem.** 3-SAT $\leq_p$ IND-SET
- Given an instance Φ of 3-SAT, we construct an instance $\langle G, k \rangle$ of IND-SET s.t. G has an independent set of size k iff ϕ is satisfiable.

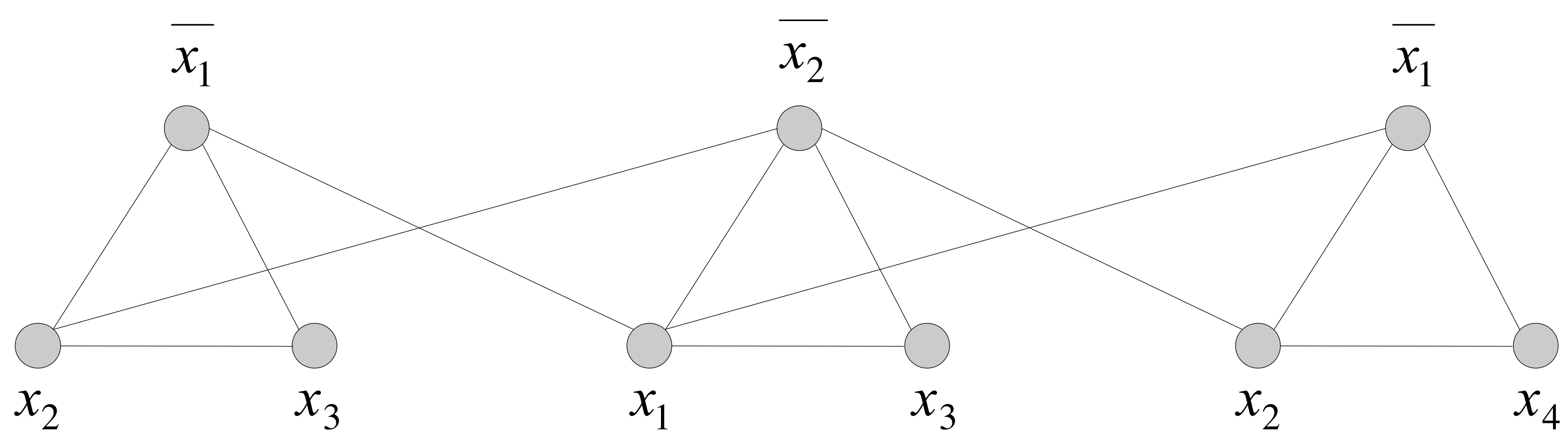


Map the Problems

3SAT	What is a possible solution?	**Ind-Set**
An assignment of T/F to variables		A selection of vertices to be an IS S
	What is the requirement?	
Each clause must contain at least one literal that is True		S must contain at least k vertices
	What are the restrictions?	
x can be true iff $\bar{x}$ is assigned false		If $(u, v) \in E$, then both u and v cannot be in S

$3SAT \leq_p$ IND-SET

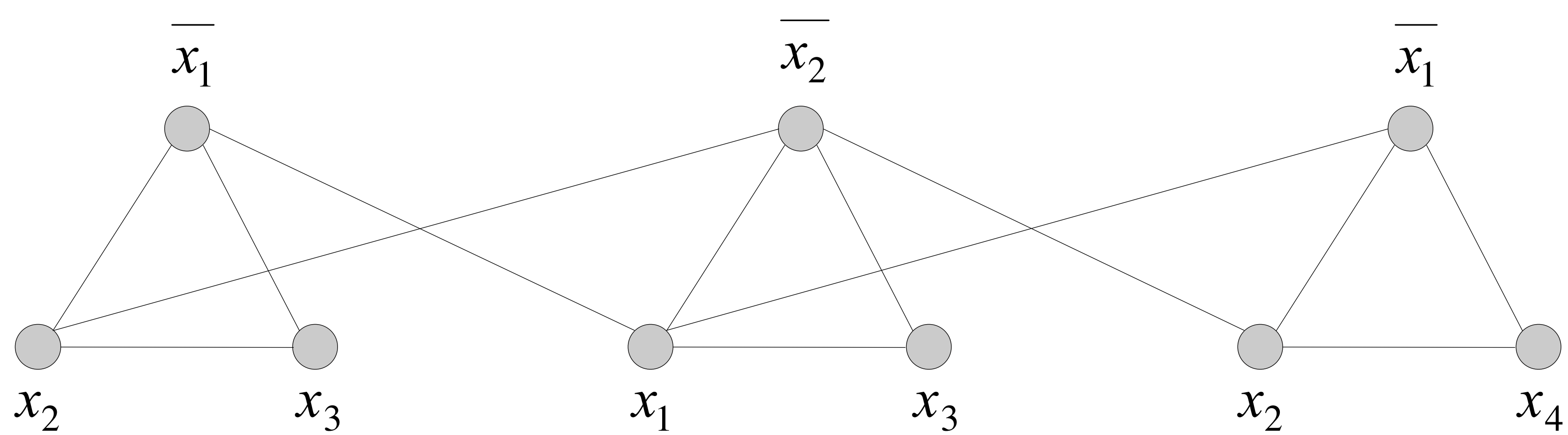
- **Reduction.** Let k be the number of clauses in Φ.
  - G has $3k$ vertices, one for each literal in Φ
  - (Clause gadget) For each clause, connect the three literals in a triangle
  - (Variable gadget) Each variable is connected to its negation



$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3SAT$\leq_p$ IND-SET

- **Observations.**
    - Any independent set in G can contain at most 1 vertex from each clause triangle
    - Only one of x_i or $\overline{x_i}$ can be in an independent set (*consistency*)



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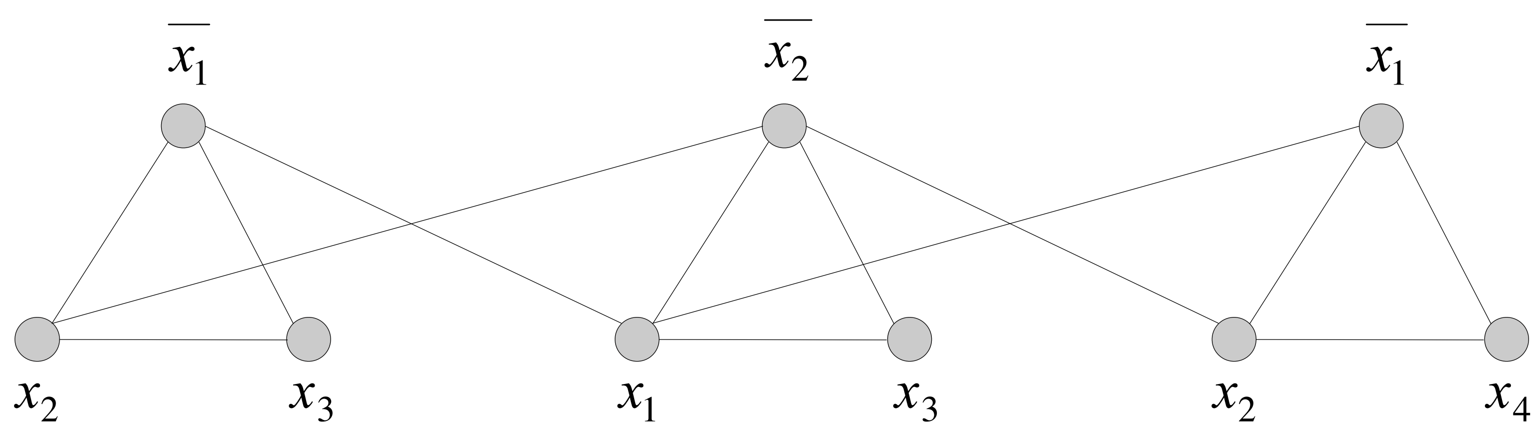
3SAT $\leq_p$ IND-SET

- **Claim.** Φ is satisfiable iff G has an independent set of size k
- ($\Rightarrow$) Suppose Φ is satisfiable, consider a satisfying assignment
    - There is at least one true literal in each clause
    - Select one true literal from each clause/triangle
    - This is an independent set of size k



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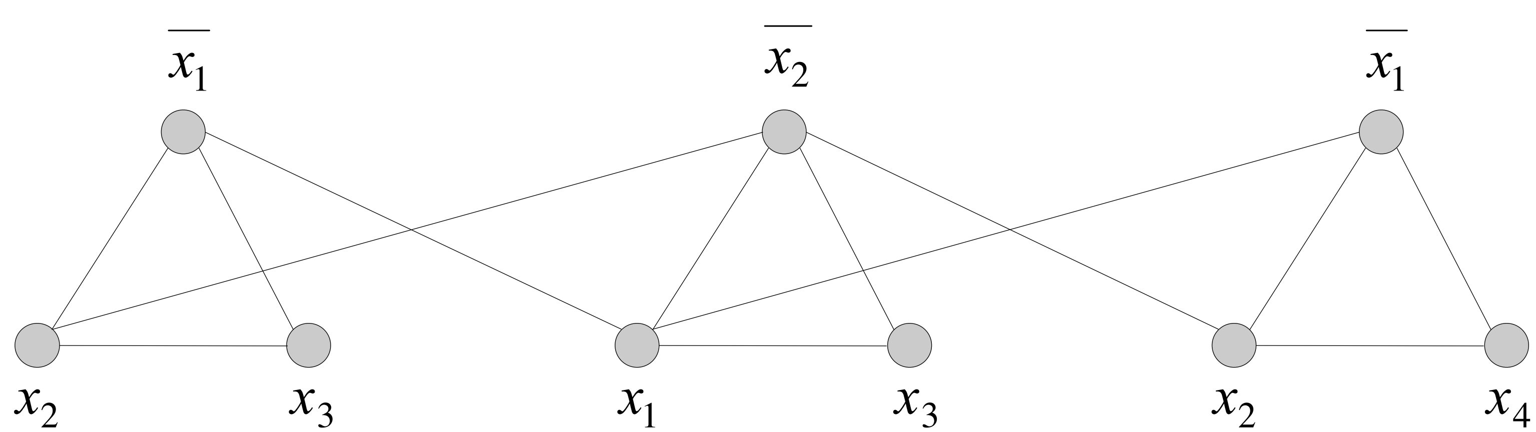
3SAT $\leq_p$ IND-SET

- **Claim.** Φ is satisfiable iff G has an independent set of size k
- $(\Leftarrow)$ Let S be in an independent set in G of size k
  - S must contain exactly one node in each triangle
  - Set the corresponding literals to *true*
  - Set remaining literals consistently
  - All clauses are satisfied — Φ is satisfiable ∎



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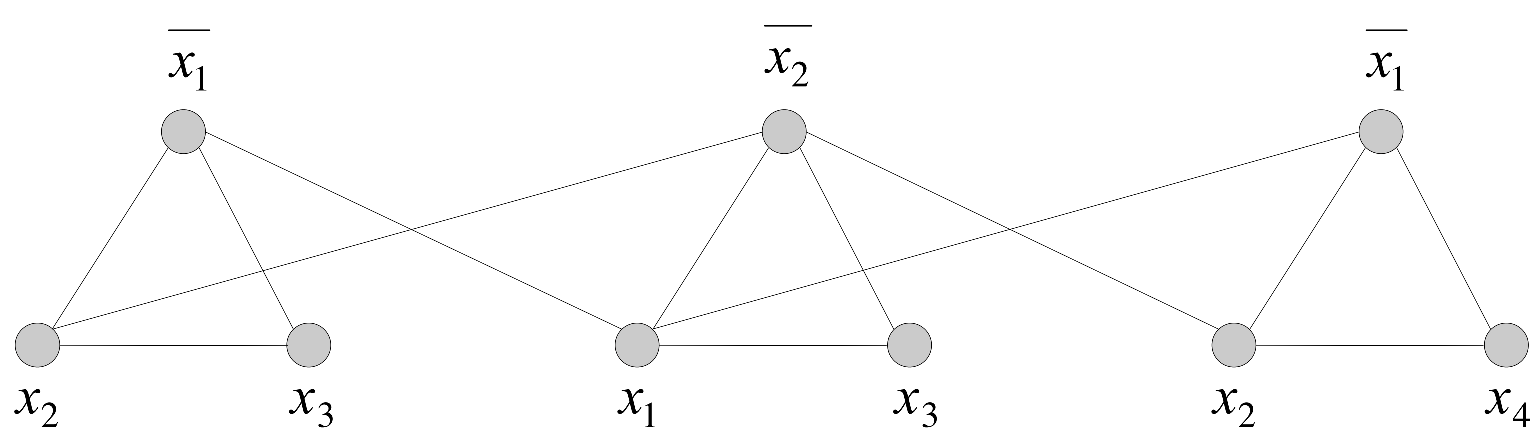
3SAT $\leq_p$ IND-SET

- Our reduction is clearly polynomial time in the input
    - G has 3k nodes, where k is #clauses, and n edges (one for each variable in G)
- Since independent set is in NP (shown previously)
    - Independent set is NP complete



$$\Phi = \left(\overline{x_1} \lor x_2 \lor x_3\right) \land \left(x_1 \lor \overline{x_2} \lor x_3\right) \land \left(\overline{x_1} \lor x_2 \lor x_4\right)$$

Reduction Strategies

- Equivalence
  - **VERTEX-COVER** $\equiv_p$ **IND-SET**
- Special case to general case
  - **VERTEX-COVER** $\leq_p$ **SET-COVER**
- Encoding with gadgets
  - **3-SAT** $\leq_p$ **IND-SET**
- Transitivity
  - **3-SAT** $\leq_p$ **IND-SET** $\leq_p$ **VERTEX-COVER** $\leq_p$ **SET-COVER**
  - Thus, **IND-SET**, **VERTEX-COVER** and **SET-COVER** are NP hard
  - Since they are all in NP, also NP - complete

List of NPC Problems So Far

- **3-SAT**
- **INDEPENDENT SET**
- **VERTEX COVER**
- **SET COVER**
- **CLIQUE**
- More to come:
    - **Subset Sum**
    - **Knapsack**
    - **3-COLOR**
    - **Hamiltonian cycle / path**
    - **TSP**

Steps to Prove X is NP Complete

- Step 1. Show X is in **NP**
- Step 2. Pick a known NP hard problem Y from class
- Step 3. Show that $Y \leq_p X$
    - Show both sides of reduction are correct: if and only if directions
    - State that reduction runs in polynomial time in input size of problem Y

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)