Algorithms: Max Flow

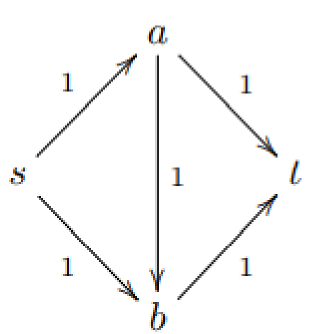
1 Consider the graph in Model 1 and the algorithm below. The algorithm claims to find in a graph V, E the maximum flow $f(e)\ \forall\ e \in E$ through the graph. On line 3, it arbitrarily chooses an unsaturated path and adds flow to it. What is an ordering of choices that enables this algorithm to successfully find a maximum flow?

Learning objective: Students will derive an efficient algorithm for finding the flow of maximum value in a graph.

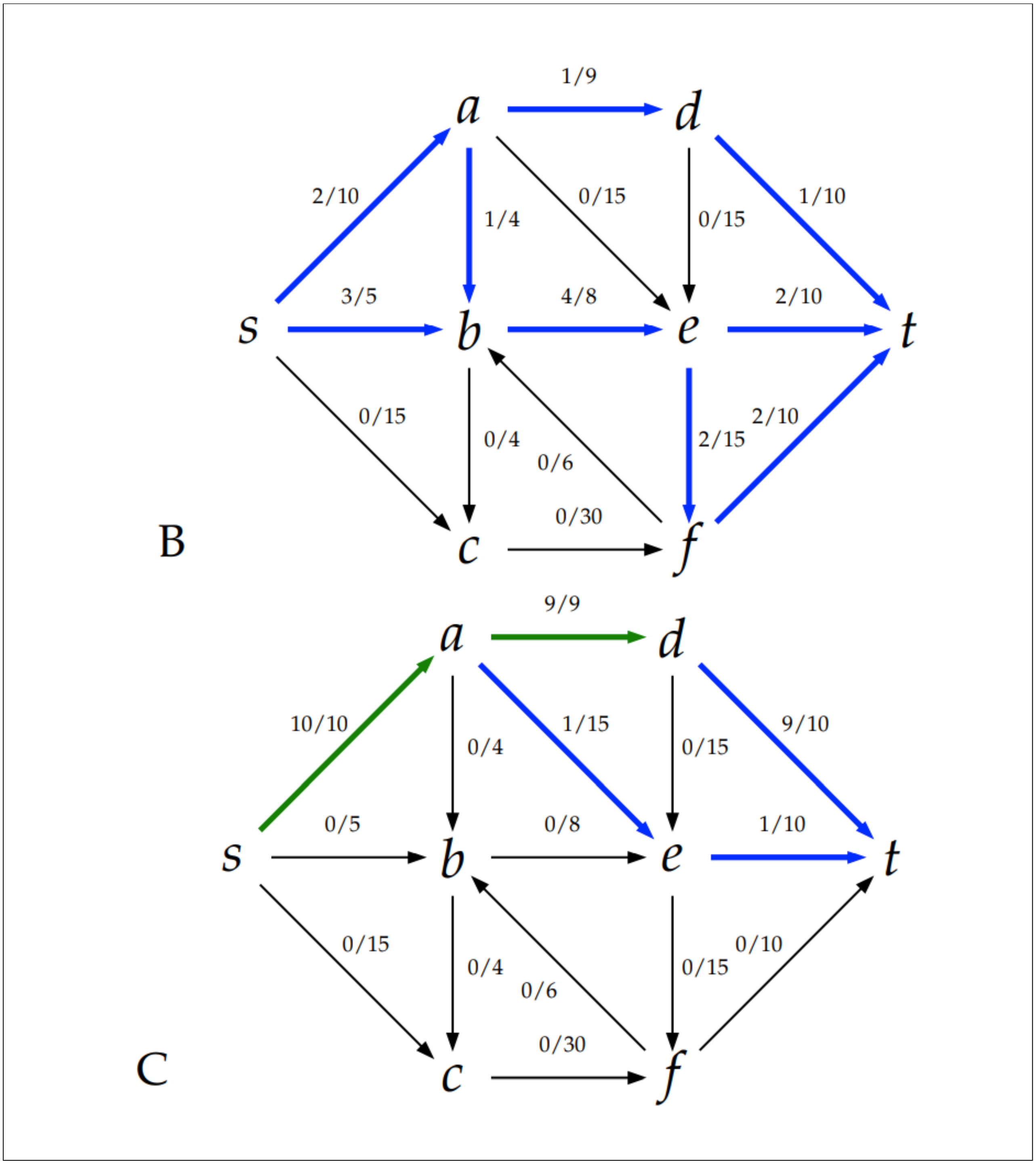
1. Initialize $f(e) \leftarrow 0$ for all $e \in E$
2. repeat
3. Find an unsaturated path P from s to t
4. $a \leftarrow$ minimum excess capacity $c(e) - f(e)$ among all edges $e \in P$
5. $f(e) \leftarrow f(e) + a$ for each edge $e \in P$
6. until no more unsaturated $s \leftarrow t$ paths

2 What is an ordering of choices that prevents this algorithm from finding a maximum flow?

Model 1: Graph



Model 2: Flow Graphs



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Definition 1. Given a network $G = (V, E)$ and a flow f on G, we define the *residual graph* G_f as $G_f = (V, E_f)$, with

$$E_f = \{e \in E \mid c(e) - f(e) > 0\} \cup \{e^R \mid e \in E, f(e) > 0\}$$

(where e^R denotes the reverse of edge e, i.e., if $e = (u, v)$, then $e^R = (v, u)$) and we define the capacities of edges in E_f by

- $c_f(e) = c(e) - f(e)$, and
- $c_f(e^R) = f(e)$.

3 Based on the given definition of a residual graph, draw a residual graph for each of the flow graphs depicted in Model 2.

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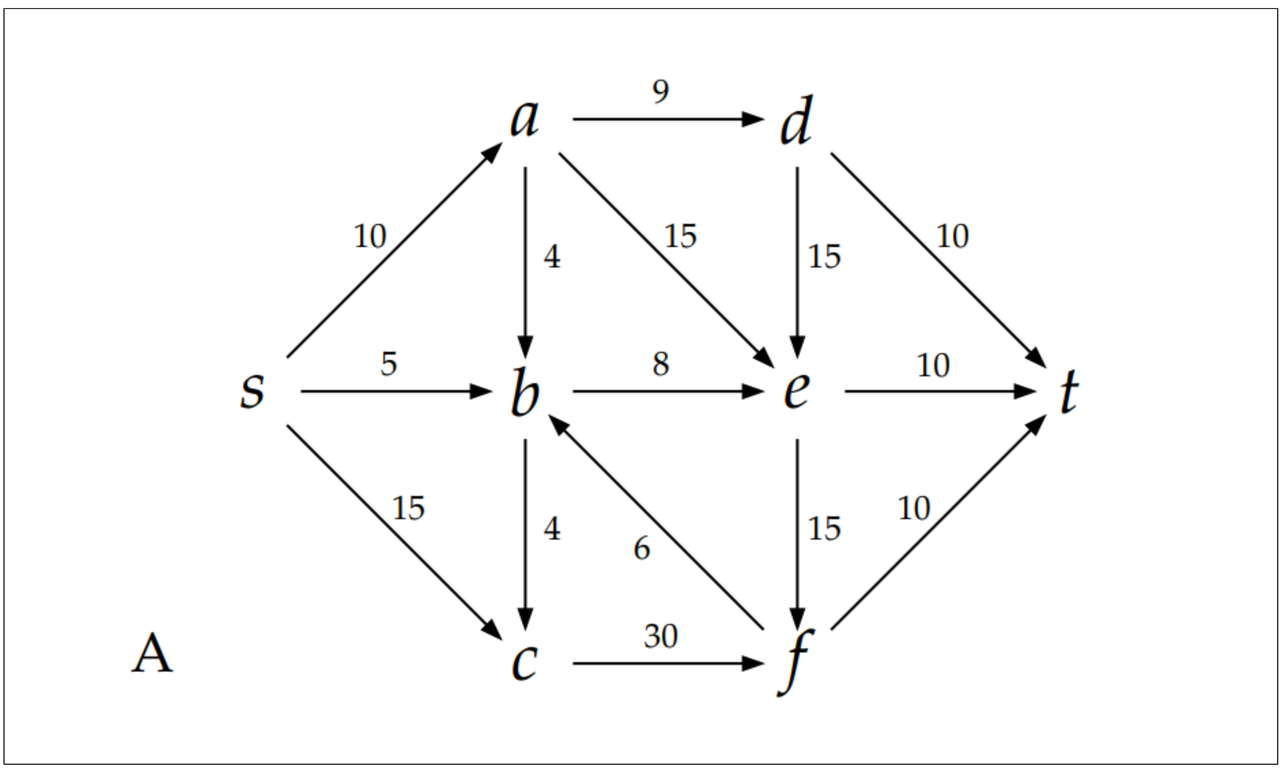
4 Consider the algorithm below.

1. Initialize $f(e) \leftarrow 0$ for all $e \in E$
2. repeat
3. $\qquad \alpha \leftarrow \min\{c_f(e) \,|\, e \in P\}$
4. $\qquad f(e) \leftarrow f(e) + \alpha$ for each $e \in P$ such that $e \in E$
5. $\qquad f(e) \leftarrow f(e) - \alpha$ for each $e \in P$ such that $e^R \in E$
6. until no paths exist from s to t in G_f

This algorithm, the Ford-Fulkerson algorithm, uses the residual graph G_f instead of the original graph to find *augmenting paths*. We execute lines 3, 4 and 5 because we have found an augmenting path in the residual graph.

Use the Ford-Fulkerson algorithm to find a flow on the graph in Model 3. Select paths in any order you like.

Model 3: Original Graph



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5 Is the flow you computed in the previous question the maximum possible flow? Why or why not?

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