"Those who cannot remember the past are condemned to repeat it."

~~— *Jorge Agustín Nicolás Ruiz de Santayana y Borrás*~~

Dynamic programming

Slow Recursion: Fibonnacci

Definition. Fibonacci numbers are defined by the following recurrence:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

Recall three different implementations of Fibonacci from our activity on Wednesday:

- Naively recursive
- Local array to “memoize” the first n numbers
- Global array, worked backwards from n

Slow Recursion: Fibonnacci

The naive recurrence was horribly *sloooooow*

```
RecFibo(n):
    if n = 0
        return 0
    else if n = 1
        return 1
    else
        return RecFibo(n − 1) + RecFibo(n − 2)
```

- **Practice:** can we lower bound the cost?
    - Step 1: Write the recurrence

$$T(n) = T(n-1) + T(n-2) + O(1)$$

Slow Recursion: Fibonnacci

Can we lower bound the running time using techniques we already know?

$$T(n) = T(n-1) + T(n-2) + \Theta(1)$$

- If we want to show that $a \geq c$, we can show $a \geq b$ and $b \geq c$

$$T(n) \geq 2T(n-2) + \Omega(1)$$

We know $T(n-1) \geq T(n-2)$

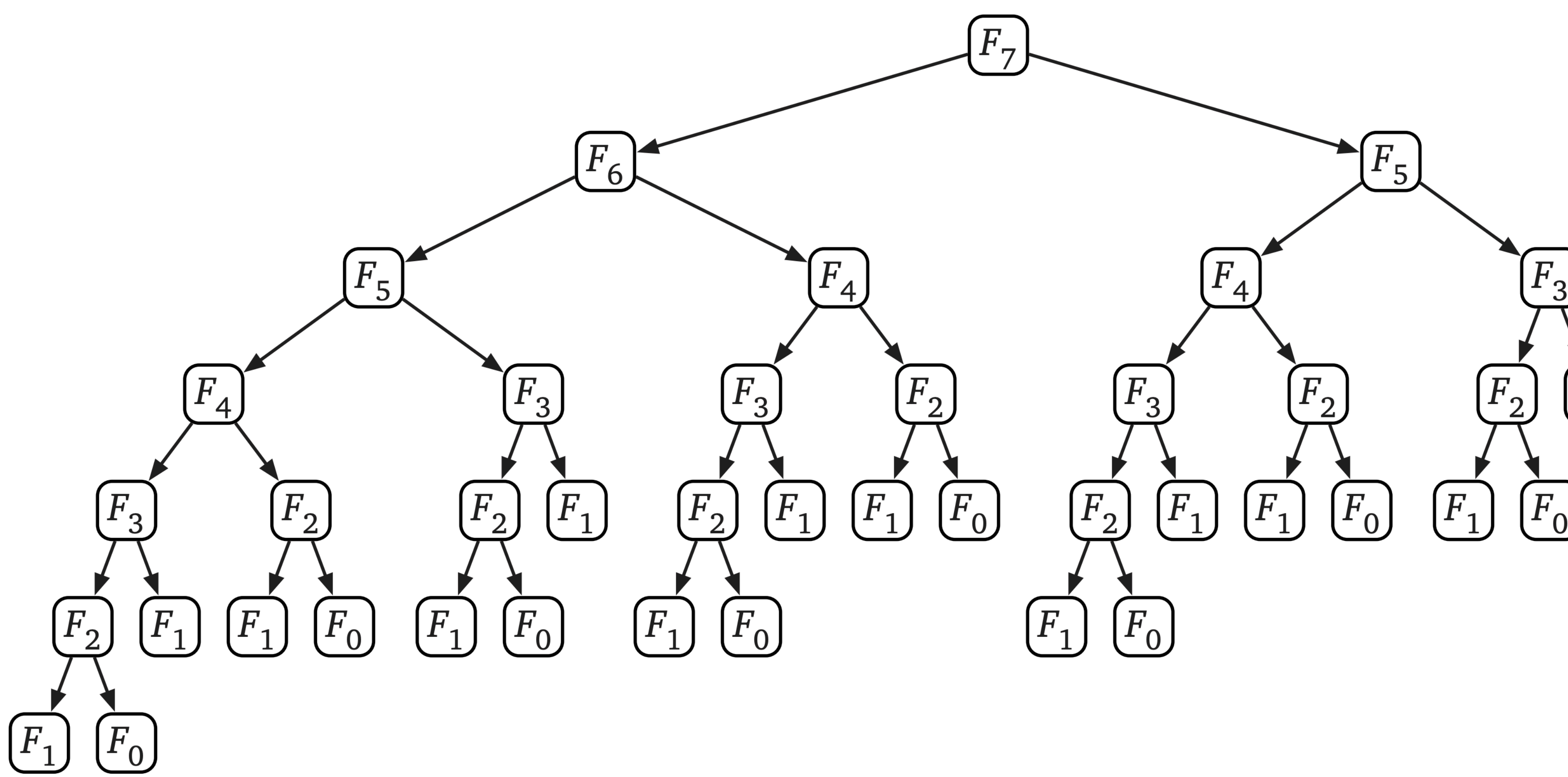
Let's draw this tree!

- There are $n/2$ levels, each level has 2^i nodes
- Level i has cost $\Omega(2^i)$

$$T(n) = \Omega(2^{n/2})$$

Memo(r)ization

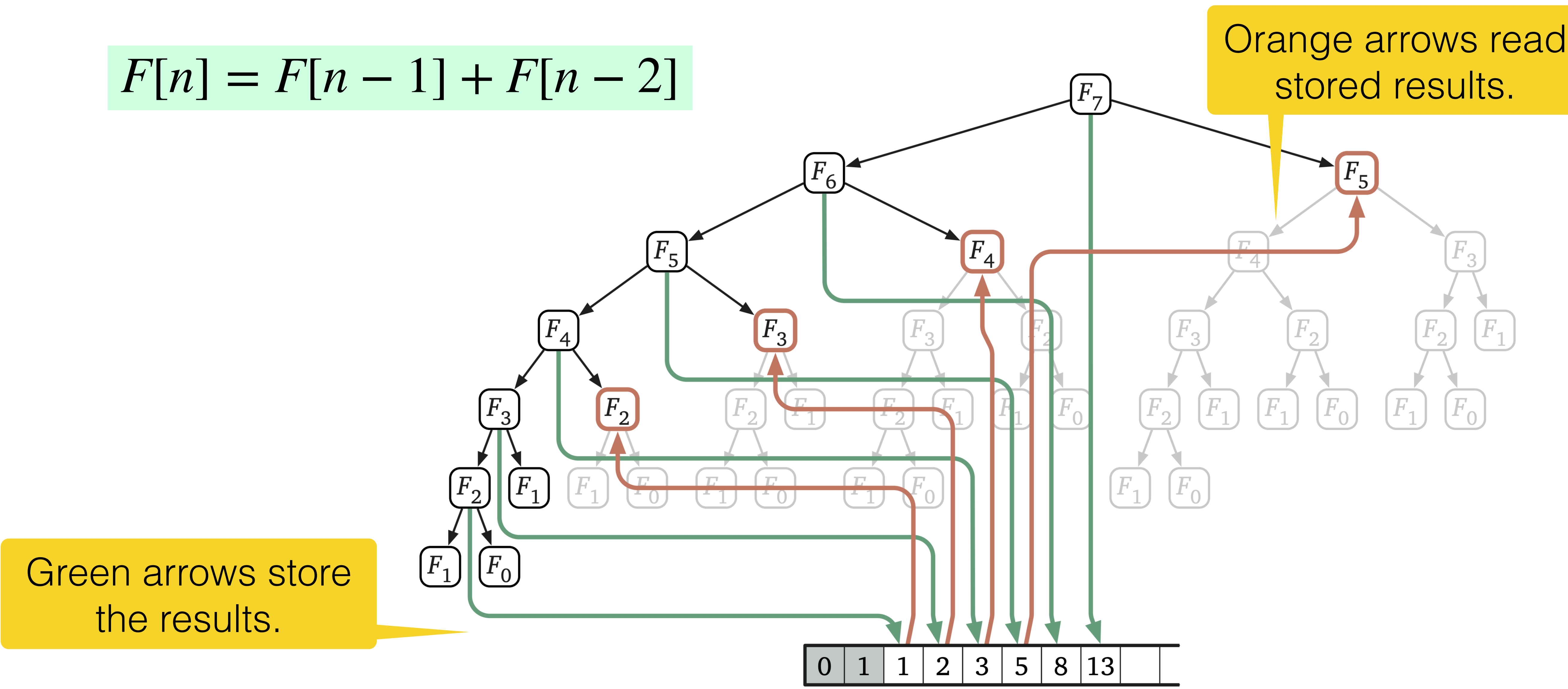
- Recursive Fibonacci algorithm is slow because it recomputes the same functions over and over
- We saw that we can speed it up considerably by writing down the results of our recursive calls, and looking them up when we need them later



Dynamic Programming: Smart Recursion

- Dynamic programming is all about smart recursion by using memoization
- Here (`fib3` from activity) we cut down on all useless recursive calls

$$F[n] = F[n-1] + F[n-2]$$



Dynamic Programing: Recursion + Memoization

- **Memoization**: technique of storing expensive function call results so that they can be looked up later
  - To be useful, we must carefully structure our algorithm to traverse problem space in appropriate order
  - Memoization is a core concept of dynamic programming, but also used elsewhere

Recipe for a Dynamic Program

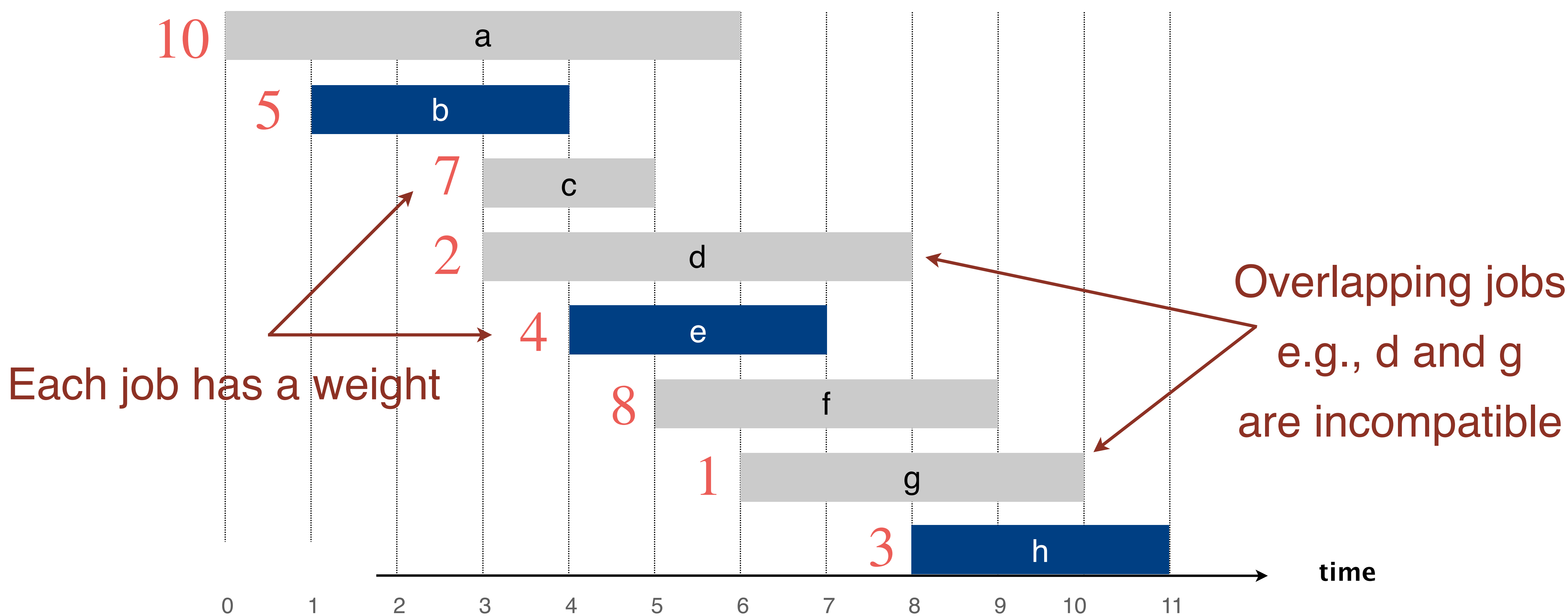
- **Formulate the right subproblem.** The subproblem must have an optimal substructure
- **Formulate the recurrence.** Identify how the results of the smaller subproblems can contribute to results of larger subproblems
- **State the base case(s)**. The subproblem(s) so small we know the answer immediately!
- **State the final answer.** (In terms of the subproblem(s))
- **Choose a memoization data structure.** Where are you going to store already computed results? (This is often a "table")
- **Identify evaluation order.** Identify the dependencies: which subproblems depend on which? Using these dependencies, identify an evaluation order
- **Analyze space and running time.** As always!

Weighted Scheduling

Further Reading: Chapter 6, KT

Weighted Scheduling

Job scheduling. Suppose you have a machine that can run one job at a time; n job requests, where each job i has a start time s_i, finish time f_i and weight $v_i \geq 0$.



Weighted Scheduling

Input. Given n intervals labeled $1,\ldots,n$ with starting and finishing times $\{(s_1, f_1), \ldots, (s_n, f_n)\}$ and non-negative weights $\{v_1, \ldots, v_n\}$.

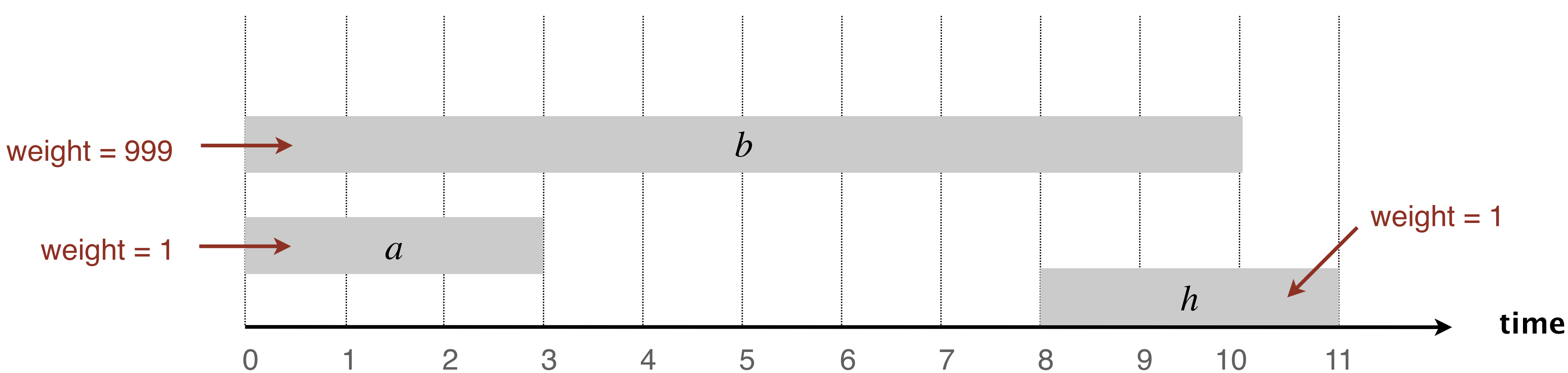
Goal. We must select compatible (non-overlapping) intervals with the maximum weight.

- That is, our goal is to find a set of intervals $I \subseteq \{1,\ldots,n\}$ that are pairwise non-overlapping and that maximize $\sum_{i \in I} v_i$

Remember Greedy?

- In Unweighted, earliest-finish-time first was optimal greedy algorithm
  - Consider jobs in order of finish times
  - Greedily pick jobs that are non-overlapping
- We proved greedy is optimal when all weights are 1
- How about the weighted interval scheduling problem?

Greedy fails spectacularly!

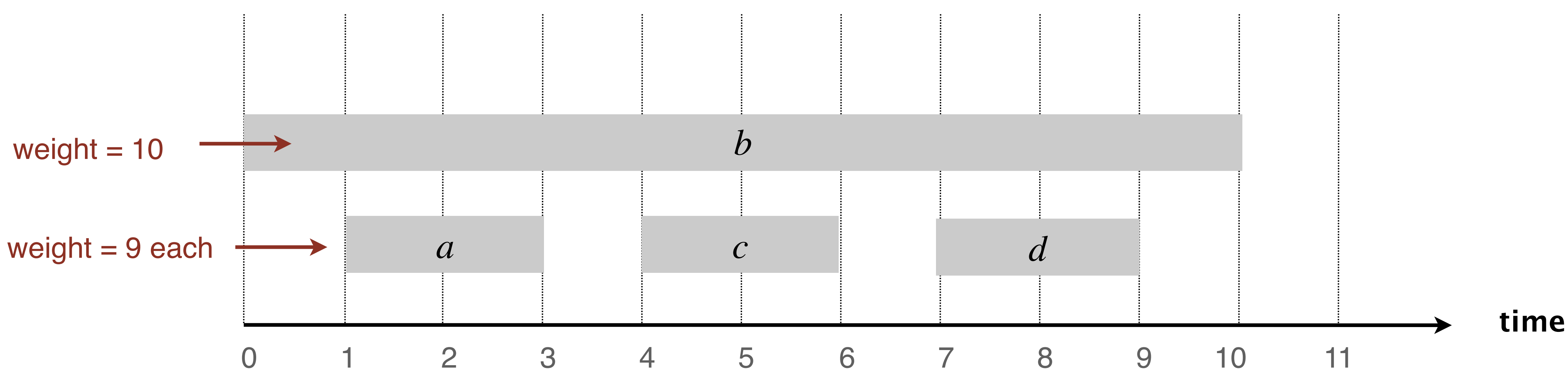


Different Greedy?

We saw that not it is important to choose the right thing to "be greedy" over. Should we just pick other optimization criteria?

- New idea: greedily select intervals with the maximum weights, remove overlapping intervals
- Does this work?

Greedy fails spectacularly!



Let's Think Recursively

The heart of dynamic programming is recursively thinking.

- Coming up with a **smaller subproblem** that has the **same optimal structure** as the original problem
- First, let's focus on the total **value** of the optimal solution, rather than the actual set of intervals. That is,
- **Optimal value:**
  The largest $\sum_{i \in I} v_i$ where intervals in I are compatible.
- Let's also define **Opt-Schedule**(n) to be the **value** of the optimal schedule that considers the first n intervals

Let's Think Recursively

Consider the last interval: *it's either in the optimal solution or it's not*.

- Whatever the optimal solution is, we can find it by considering both cases (in or out) and taking their maximum weight.
- **Case 1.** Last interval **is not** in the optimal solution
    - Remove the last interval.
      We now have a smaller subproblem!
- **Case 2.** Last interval **is** in the optimal solution
    - Anything that overlaps with this interval cannot be in the solution. Remove them.
      We now have a smaller subproblem!

Formalize the Subproblem

Opt-Schedule(i): value of the optimal schedule that only considers intervals $\{1,\ldots,i\}$, for $0 \leq i \leq n$

Base Case & Final Answer

Opt-Schedule(i): value of the optimal schedule that only considers intervals $\{1,\ldots,i\}$, for $0 \leq i \leq n$

Base Case. Opt-Schedule$(0) = 0$

Goal (Final answer.) Opt-Schedule(n)

Recurrence

How do we go from one subproblem to the next?

- The recurrence describes how to compute Opt-Schedule(i) by using values of Opt-Schedule(j) where $j < i$

Case 1. Say interval i **is not** in the optimal solution, can we write the recurrence for this case?

- Opt-Schedule$(i) =$ Opt-Schedule$(i - 1)$

Recurrence

How do we go from one subproblem to the next?

- The recurrence describes how to compute Opt-Schedule(i) by using values of Opt-Schedule(j) where $j < i$

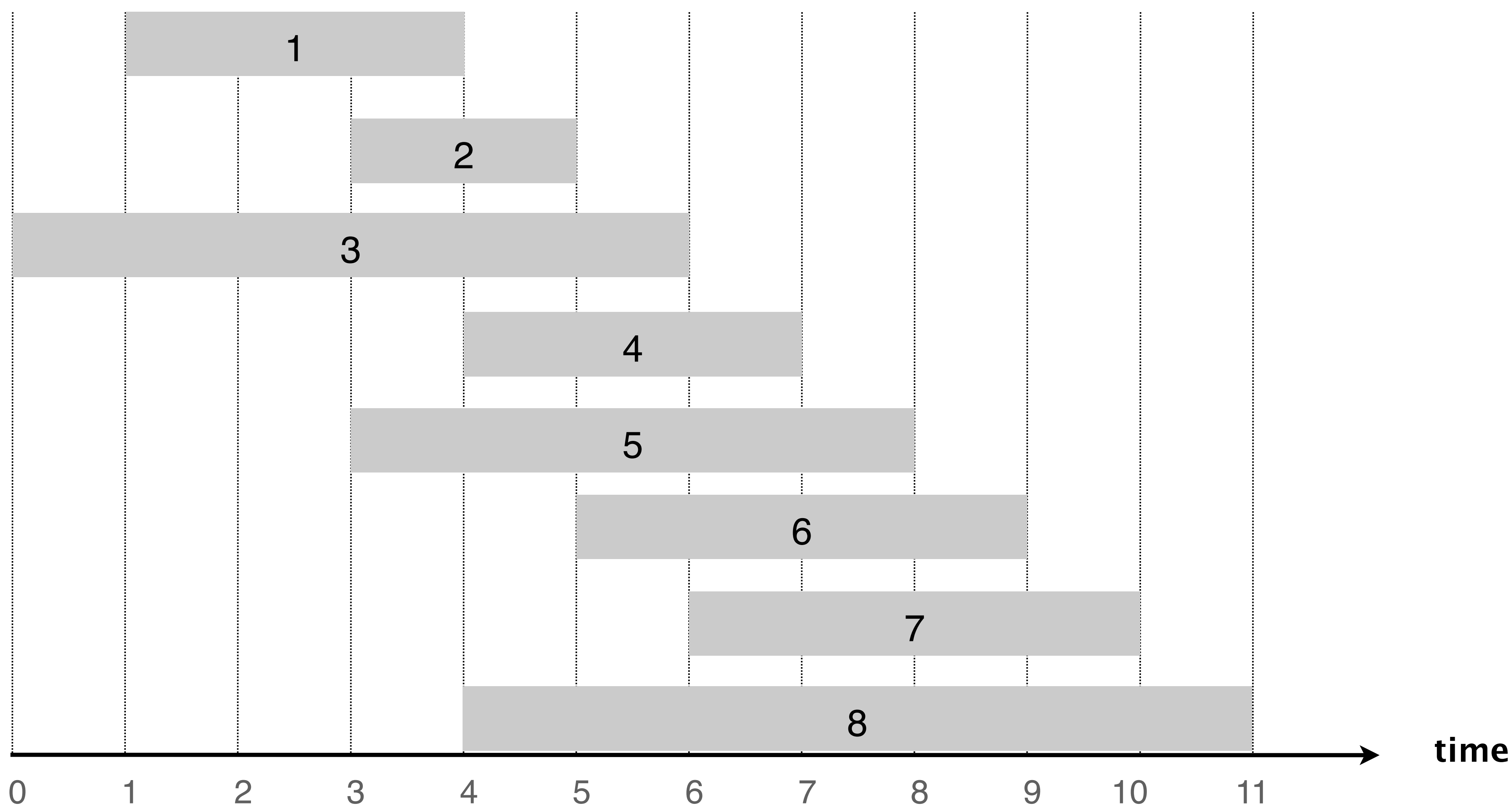
Case 2. Say interval i **is** in the optimal solution, what is the smaller subproblem we should recurse on for this case?

- No interval $j < i$ that overlaps with i can be in solution
- Need to remove all such intervals to get our smaller subproblem
- How do we do that?

Helpful Information

Suppose the intervals are sorted by finish times.

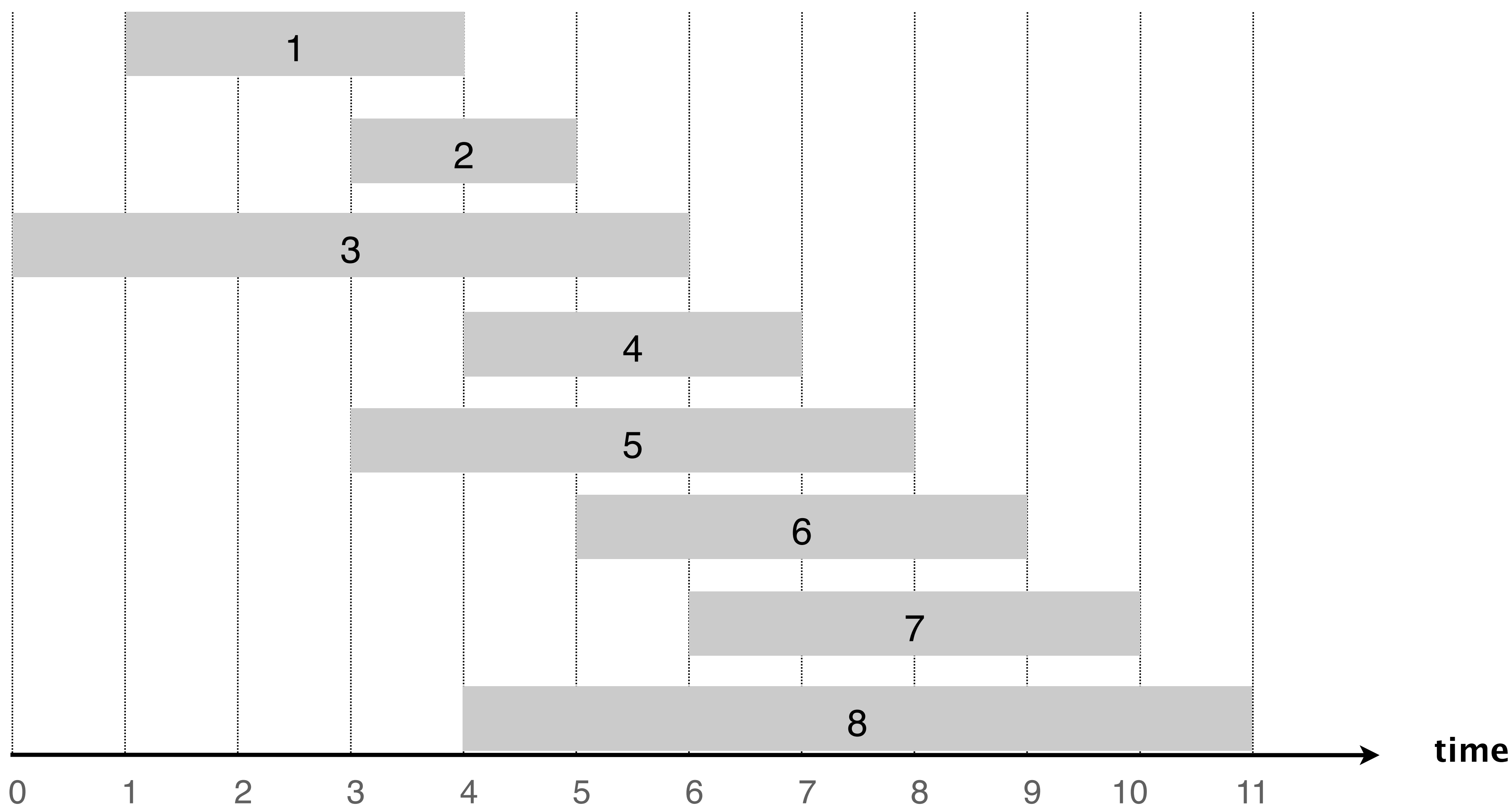
- Let $p(j)$ be the predecessor of j. That is, largest index $i < j$ such that intervals i and j are not overlapping
- Define $p(j) = 0$ if all intervals $i < j$ overlap with j



Helpful Information

Let $p(j)$ be the predecessor of j. That is, largest index $i < j$ such that intervals i and j are not overlapping.

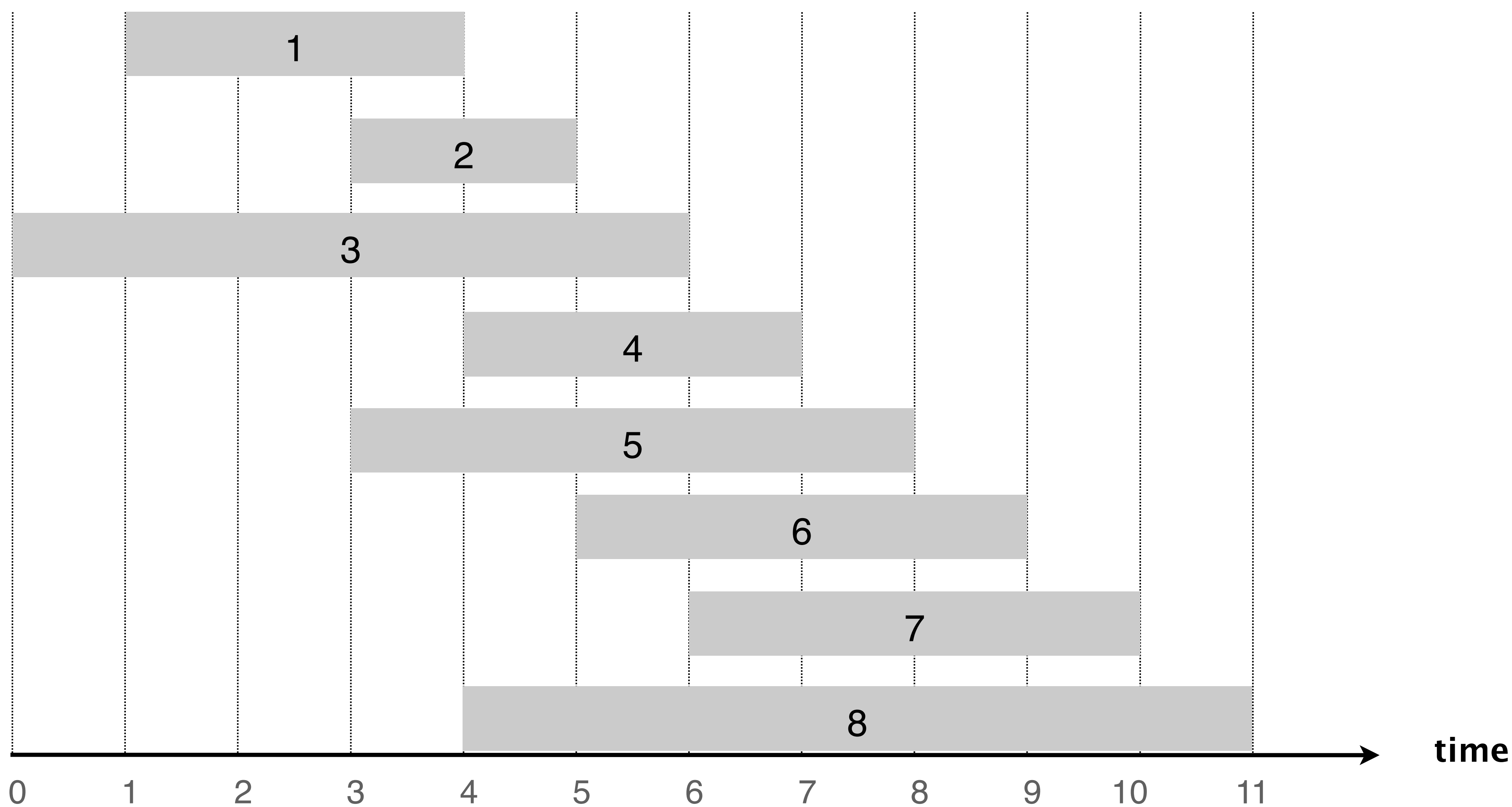
- $p(8) = ?, \;\; p(7) = ?, \;\; p(2) = ?$



Helpful Information

Let $p(j)$ be the predecessor of j. That is, largest index $i < j$ such that intervals i and j are not overlapping.

- $p(8) = 1, \;\; p(7) = 3, \;\; p(2) = 0$



Recurrence

How do we go from one subproblem to the next?

- The recurrence describes how to compute Opt-Schedule(i) by using values of Opt-Schedule(j) where $j < i$

Case 2. Say interval i **is** in the optimal solution, what is the smaller subproblem we should recurse on for this case?

- Suppose we know $p(i)$ (the predecessor of i), how can we write the recurrence for this case?
- Opt-Schedule$(i) =$ Opt-Schedule$(p(i)) + v_i$

DP Recurrence

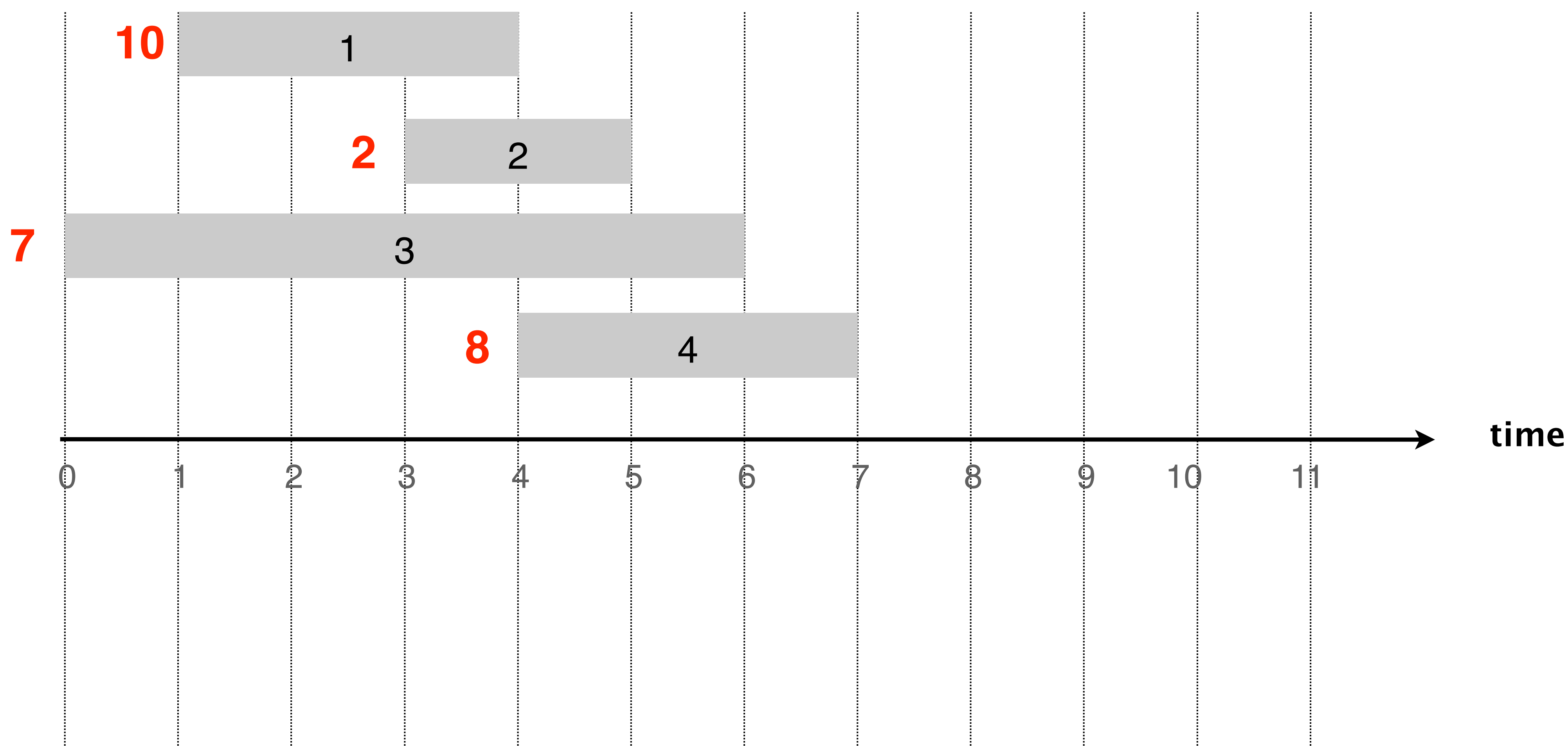
$$\text{Opt-Schedule}(i) = \max\{\text{Opt-Schedule}(i-1),\ v_i + \text{Opt-Schedule}(p(i))\}$$

Optimal schedule that excludes interval i

Optimal schedule that includes interval i

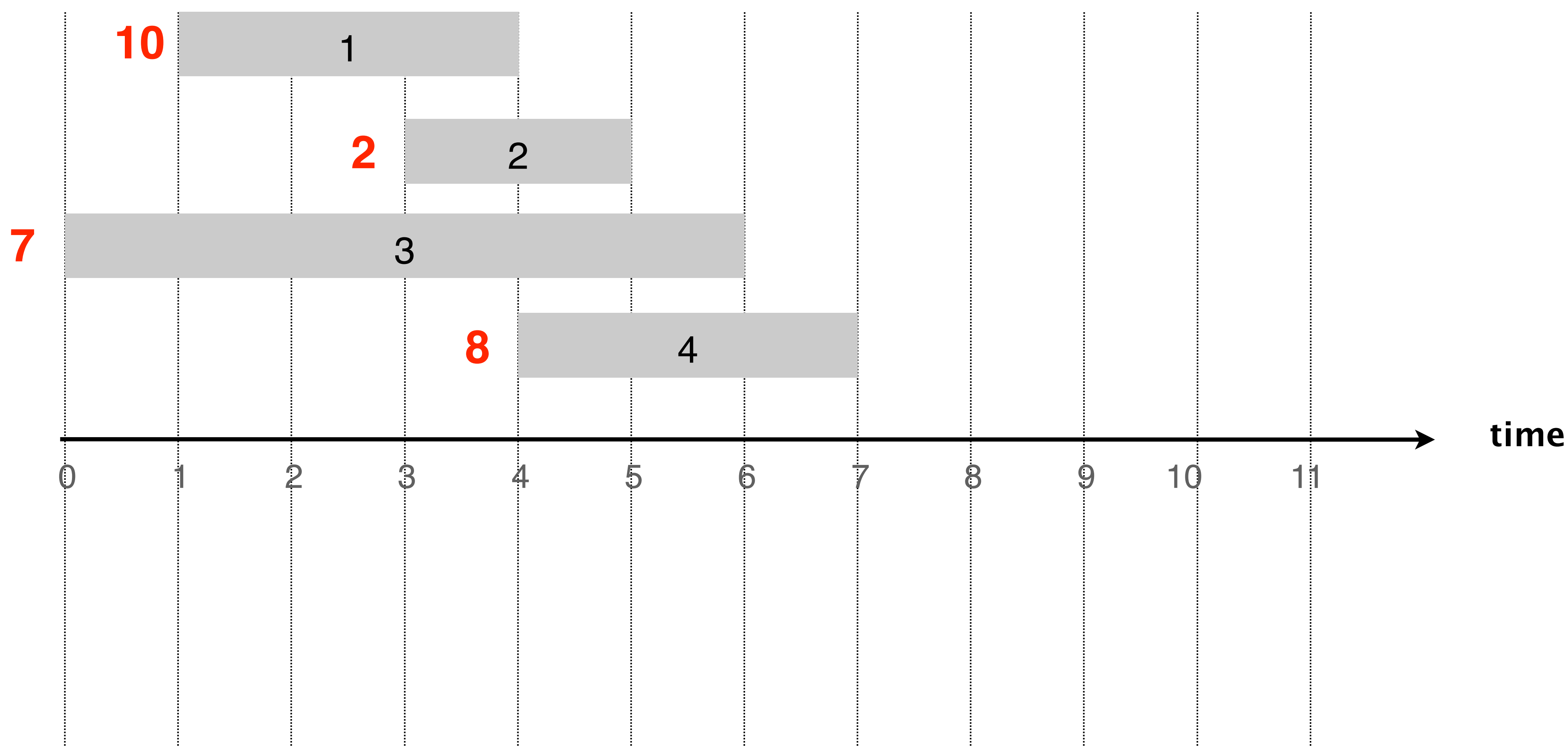
Filling Out the DP Table

0				
0	1	2	3	4



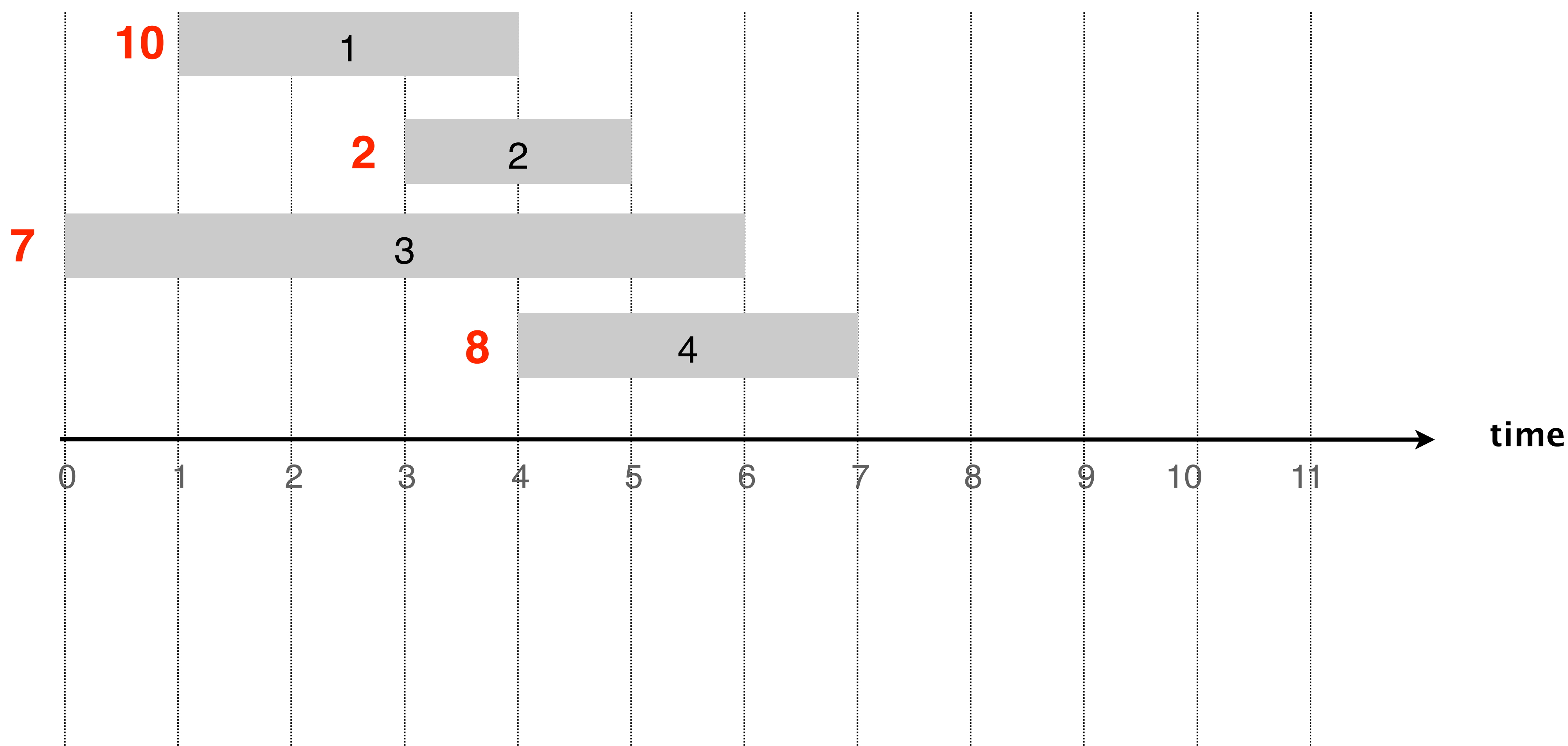
Filling Out the DP Table

0	1	2	3	4
0	10			



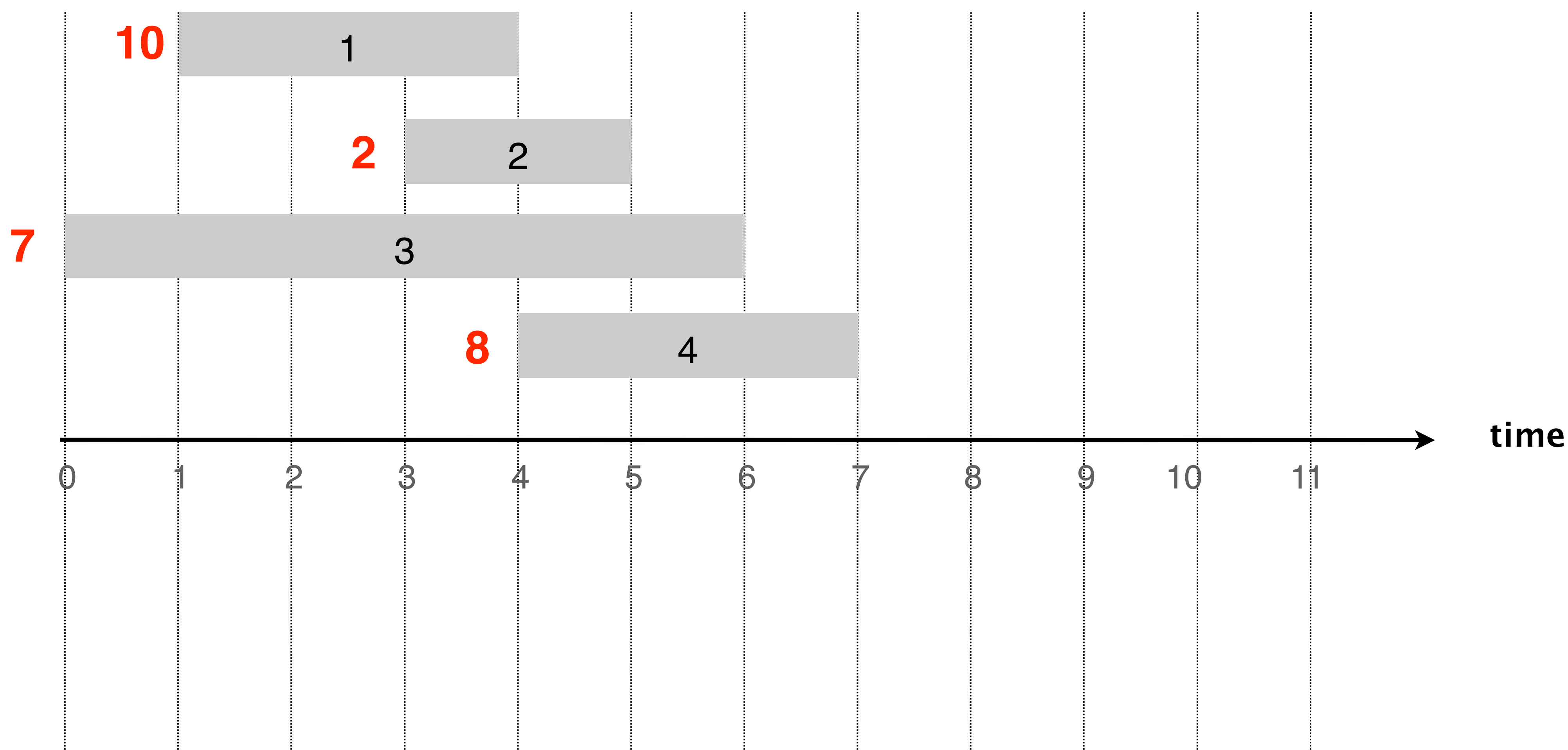
Filling Out the DP Table

0	10	10		
0	1	2	3	4



Filling Out the DP Table

0	10	10	10	18
0	1	2	3	4



Summary of DP

- **Subproblem**. Formulate the optimal substructure
  - For $0 \leq i \leq n$, let Opt-Schedule(i) be the value of the optimal schedule that only uses intervals $\{1, \ldots, i\}$
- **Recurrence.** How to go from one subproblem to the next
  - Opt-Schedule$(i) =$ $\max\{$Opt-Schedule$(i-1)$, $v_i +$ Opt-Schedule$(p(i))\}$
- **Base case**. The problem(s) we immediately know the answer to.
  - Opt-Scheduler$(0) = 0$ (no intervals to schedule)
- **Correctness**.
  - Use induction based on the recurrence

Remaining Pieces

- Final answer in terms of subproblem?
  - Opt-Schedule$[n]$
- Evaluation order (in what order can be fill the DP table)
  - $i = 0 \rightarrow n$, start with base case and use that to fill the rest
- Memoization data structure: 1-D array
- Final piece:
  - Running time and space
  - Space: $O(n)$
  - Time: preprocessing + time to fill array

Computing $p[i]$ (Preprocessing)

- How quickly can we compute $p[i]$?
  - We could do a linear scan for each i: $O(i)$ per interval
  - This would be $O(n^2)$ overall…
- What if we had intervals sorted by their finish time $F[1,\ldots,n]$
  - For each interval, we could binary search over $F[1,\ldots,n]$ to find the first $j < i$ such that $f_j \leq s_i$
  - Binary searching would take $O(\log n)$ per interval, $O(n \log n)$ total
- Time $O(n \log n)$ to compute the array $p[]$
  - This covers sorting + binary searching

Running Time

- How many subproblems do we need to solve?
  - $O(n)$
- How long does it take to solve a subproblem?
  - $O(1)$ to take the max
- Preprocessing time:
  - Need to sort; $O(n \log n)$
  - Need to find $p(i)$ for all each i: $O(n \log n)$
- Overall: $O(n \log n) + O(n) = O(n \log n)$

Wait!!! We've only computed the value, not the actual interval set!!!

Recreating Chosen Intervals

Suppose we have $M[]$ of optimal weights.

- **Big Question**: How can we reconstruct the optimal set of intervals?

Identifying which of the two cases was larger tells us whether or not interval i was included:

$$\text{Opt-Schedule}(i) =$$

$$\max\{\text{Opt-Schedule}(i-1),\ v_i + \text{Opt-Schedule}(p(i))\}$$

This value is bigger:
***i* not in OPT**

This value is bigger:
***i* is in OPT**

Recursive Solution?

Suppose for now that we do not memoize: just a divide and conquer recursion approach to the problem.

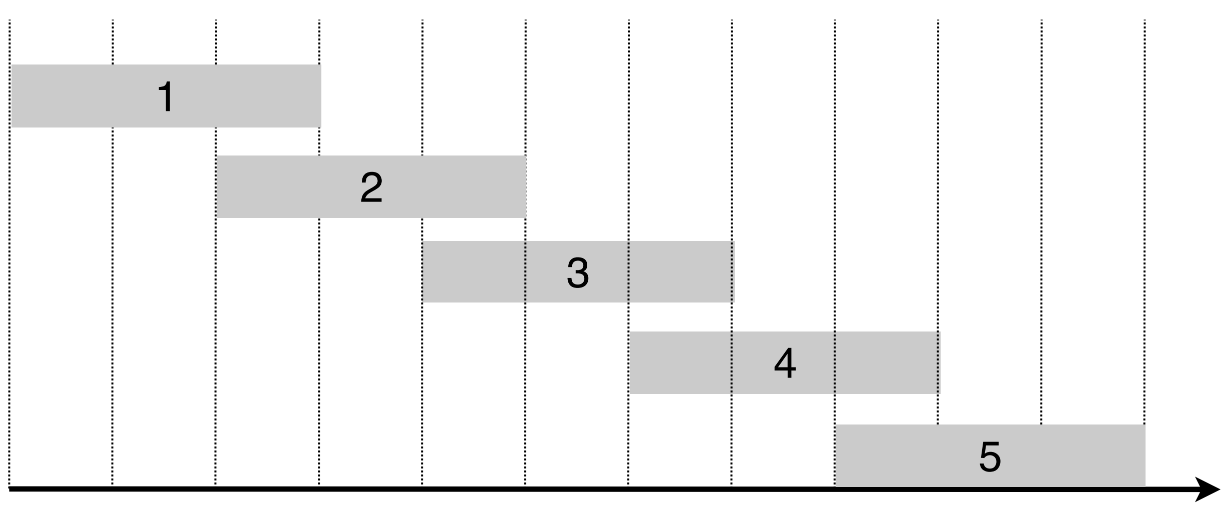
Opt-Schedule(i):

- If $j = 0$, return 0
- Else
  - Return $\max($Opt-Schedule$(j-1), v_j +$ Opt-Schedule$(p(j)))$

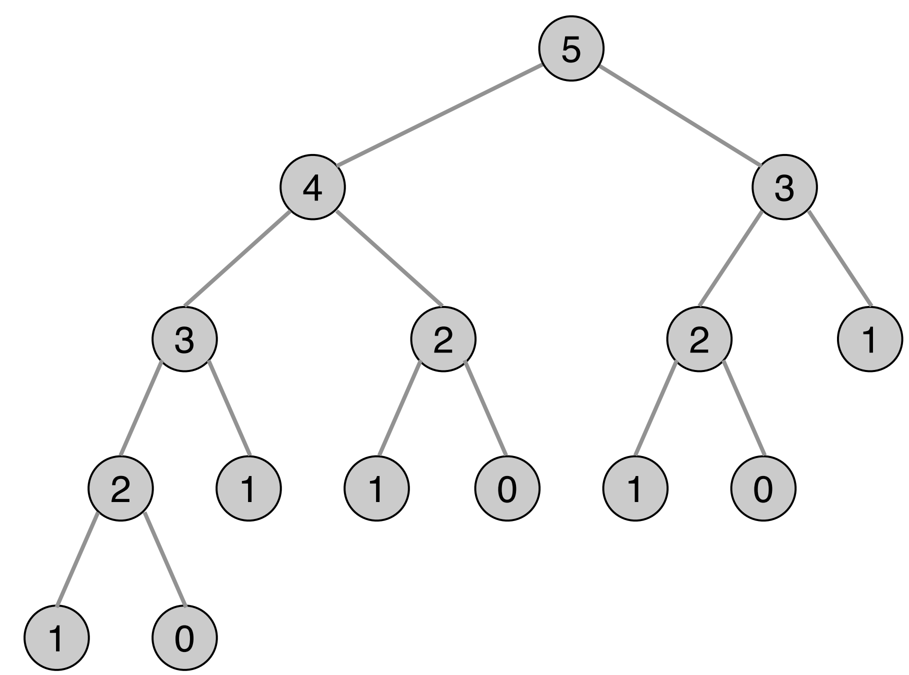
- How many recursive calls in the worst case?
  - Depends on $p(i)$
- Can we create a really bad instance?

Recursive Solution: Exponential

- For this example, asymptotically how many recursive calls?
- Grows like the Fibonacci sequence (exponential): $T(n) = T(n-1) + T(n-2) + O(1)$
- Lots of redundancy!
  - How many distinct subproblems are there to solve?
  - Opt-Schedule(i) for $1 \leq i \leq n+1$



p(1) = 0, p(j) = j-2



recursion tree

Dynamic Programming Tips

- Recurrence/subproblem is the key!
  - DP is a lot like divide and conquer, while writing extra things down
  - When coming to a new problem, ask yourself what subproblems may be useful? How can you break that subproblem into smaller subproblems?
  - Be clear while writing the subproblem and recurrence!
- In DP we usually keep track of the *cost* of a solution, rather than the solution itself

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)