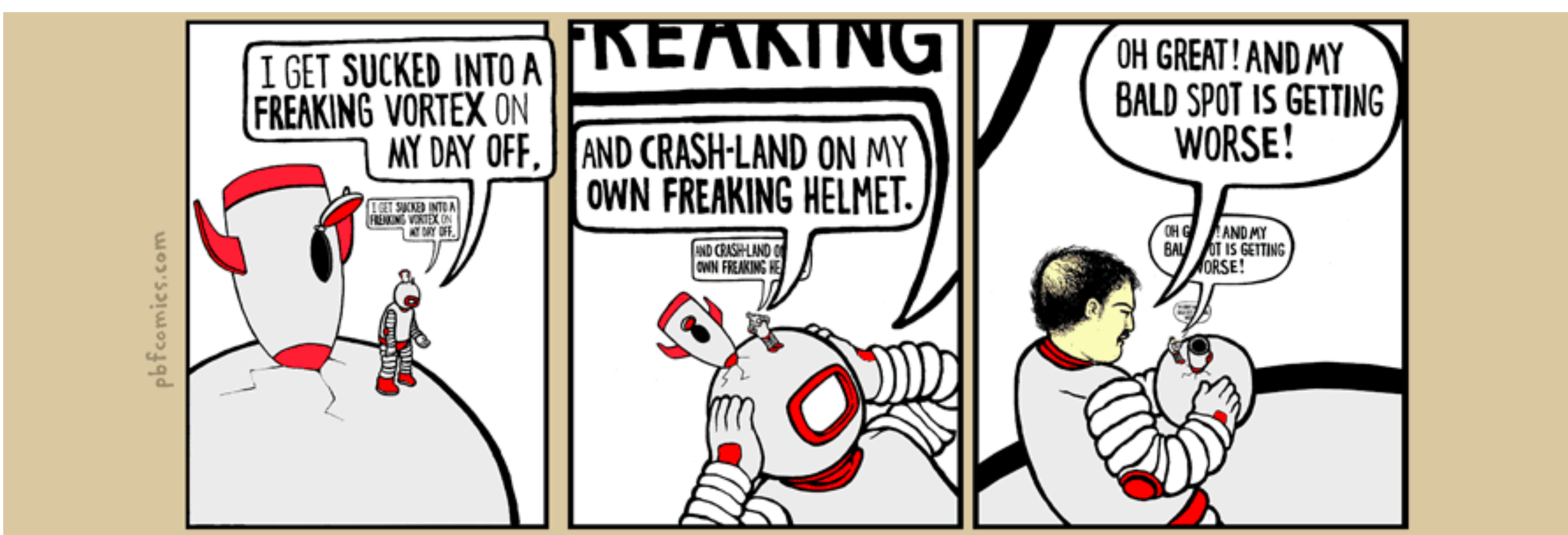
Divide and Conquer: Sorting and Recurrences

Divide & Conquer: The Pattern

- **Divide** the problem into several independent smaller instances of exactly the same problem
- **Delegate** each smaller instance to the **Recursive Leap of Faith** (technically known as induction hypothesis)
- **Combine** the solutions for the smaller instances



Review: Merge Sort

MergeSort(L):

if L has one element
return L

Base case

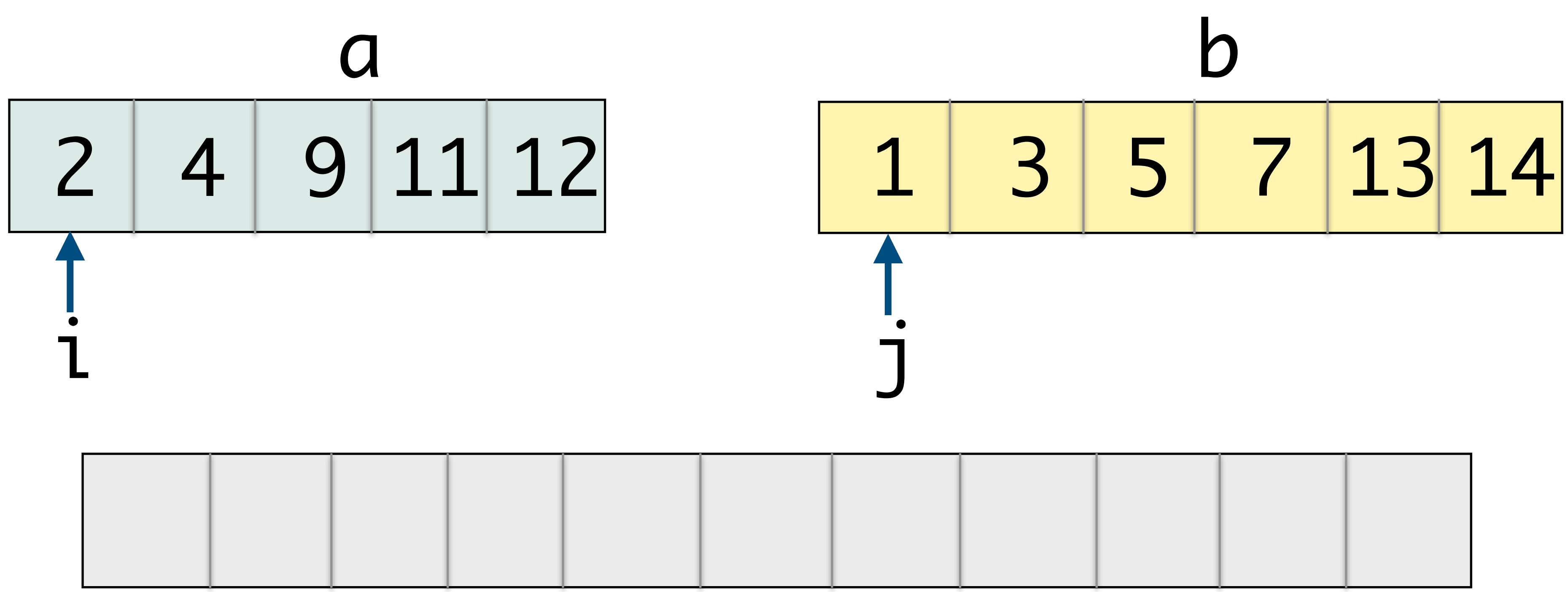
Divide L into two halves A and B
$A \leftarrow$ **MergeSort**(A)
$B \leftarrow$ **MergeSort**(B)
$L \leftarrow$ **Merge**(A, B)
return L

Recursive leaps of faith

Combine solutions

Merge Step: $\Theta(n)$

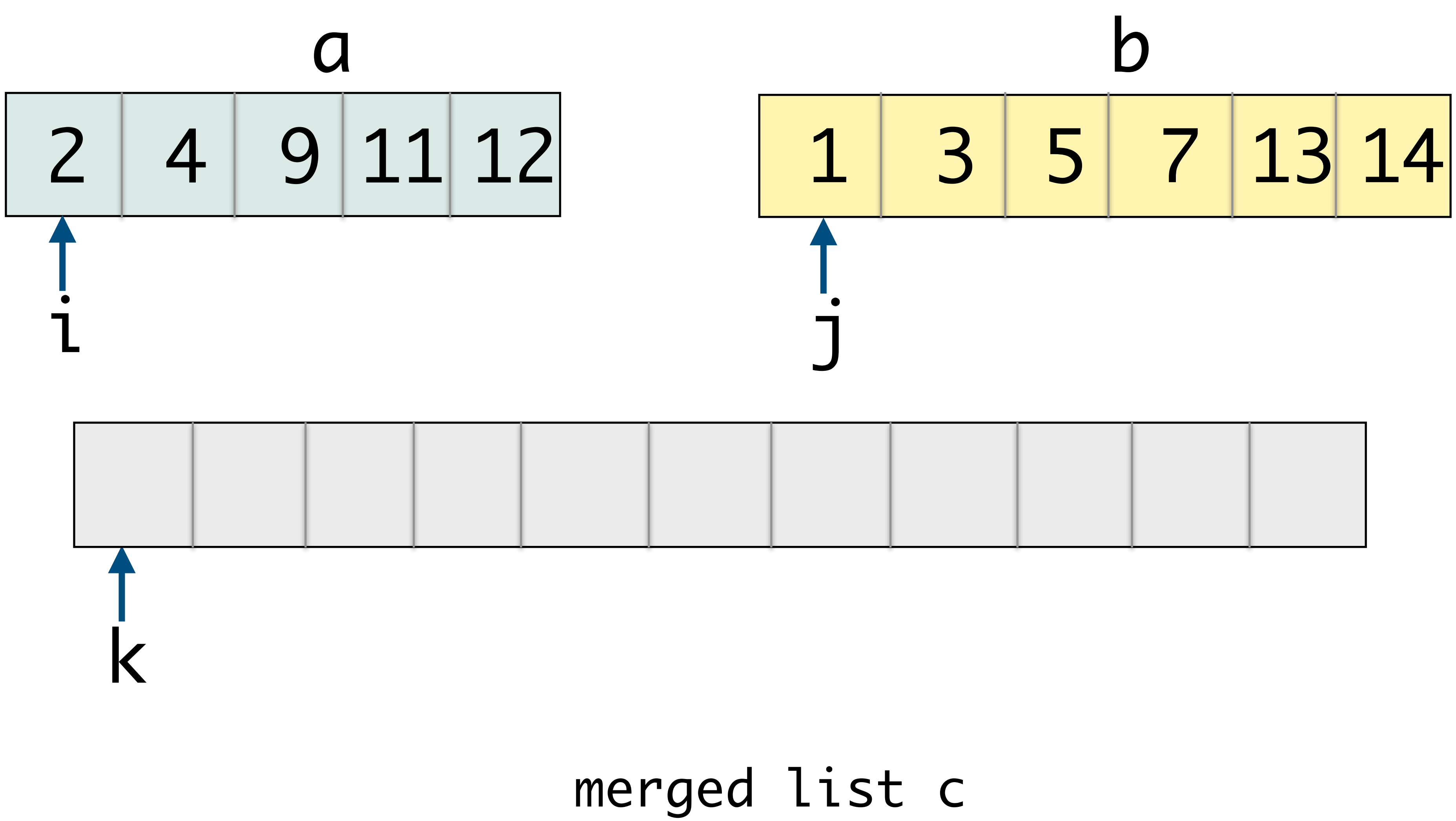
- Scan sorted lists from left to right
- Compare element by element; create new merged list



Merge Step: $\Theta(n)$

`Is a[i] <= b[j] ?`

- `Yes, a[i] appended to c, advance i`
- `No, b[j] appended to c, advance j`



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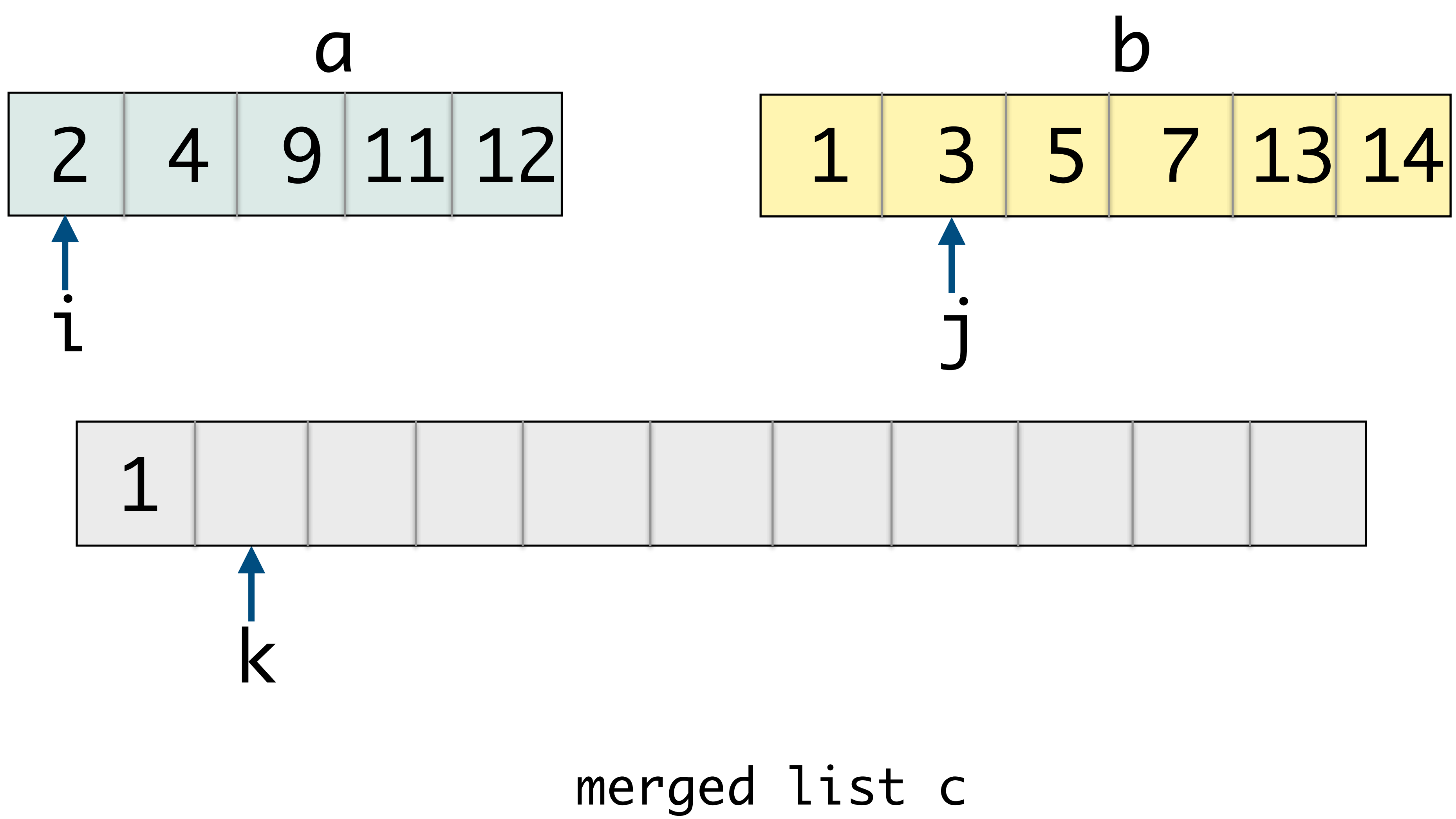

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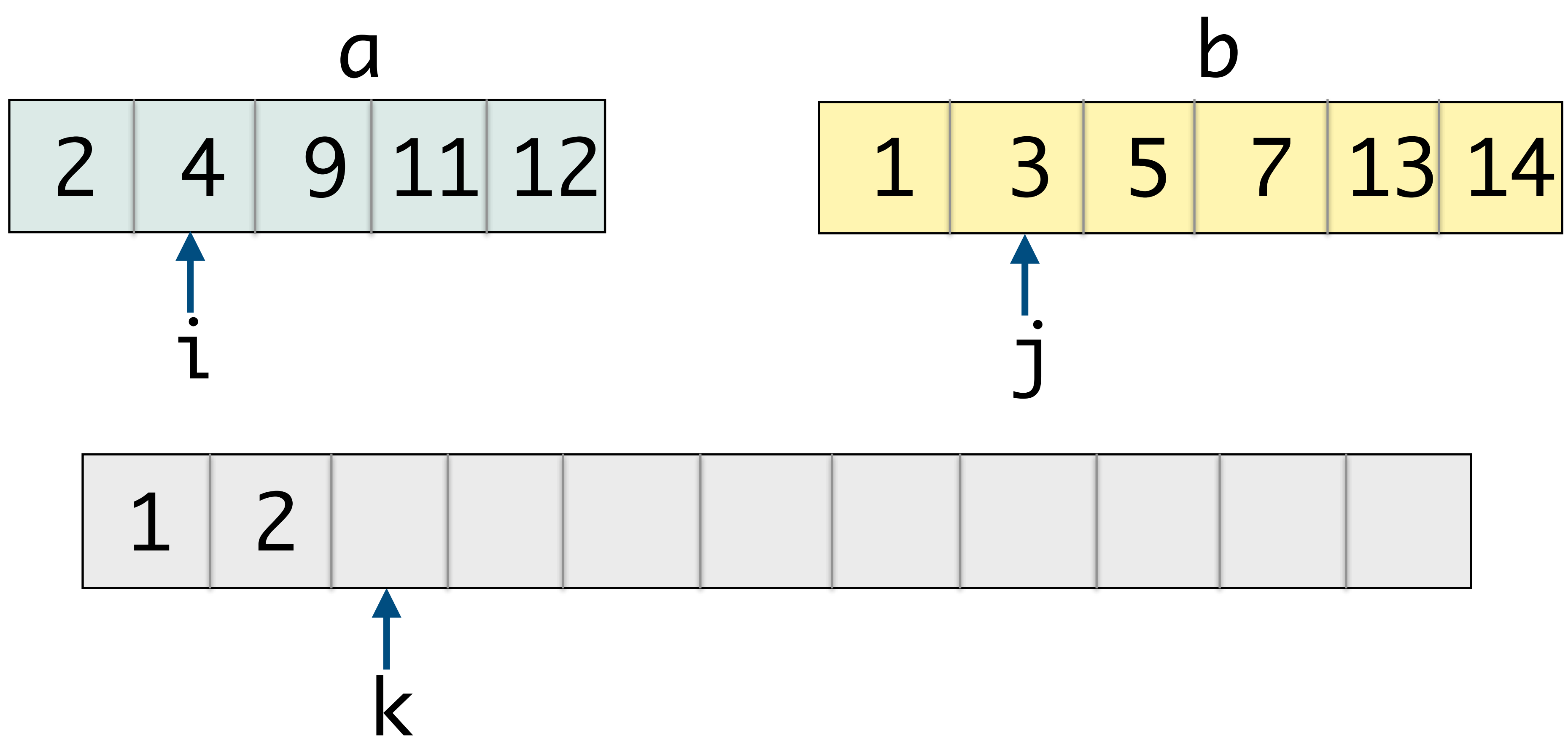

merged list c

Merge Step: $\Theta(n)$

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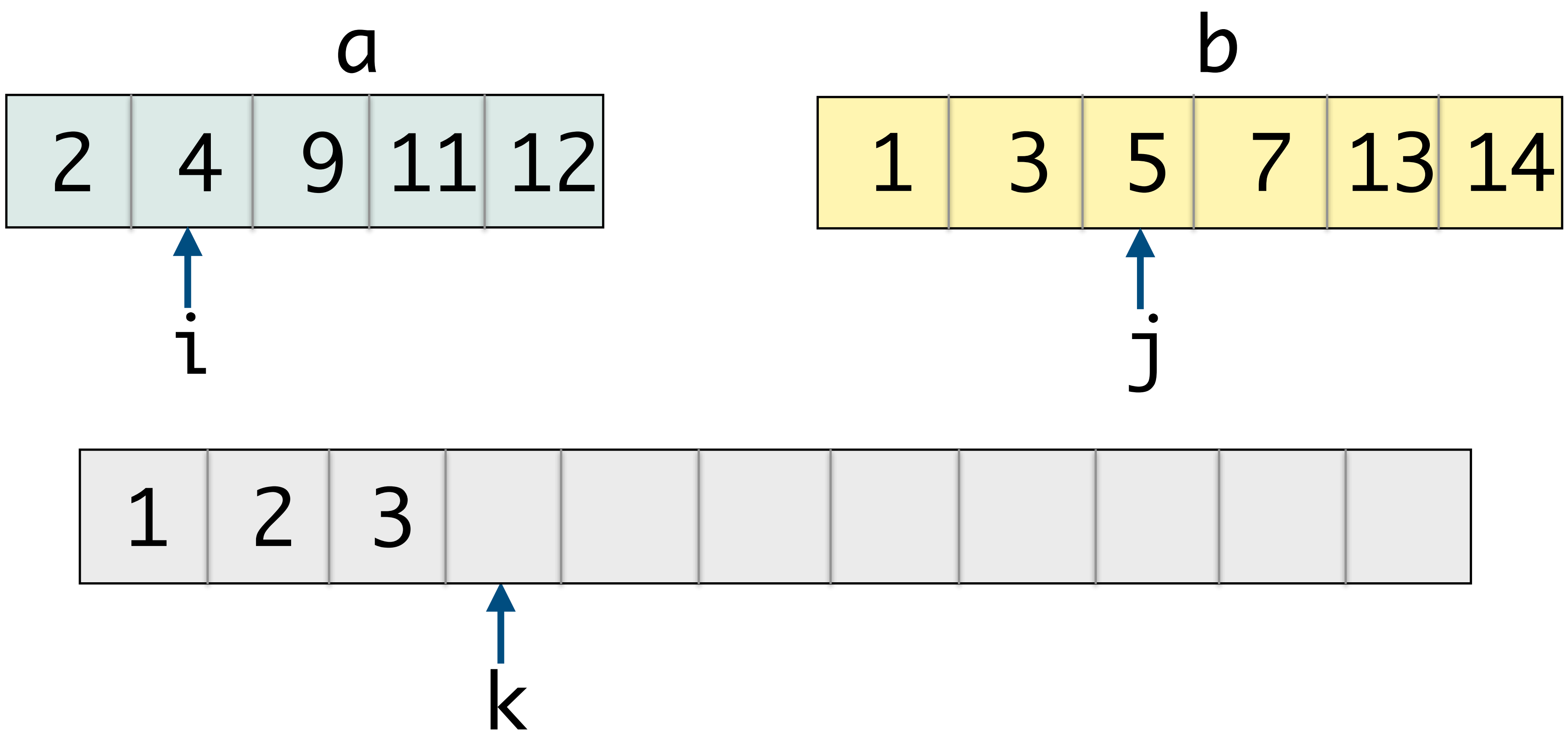

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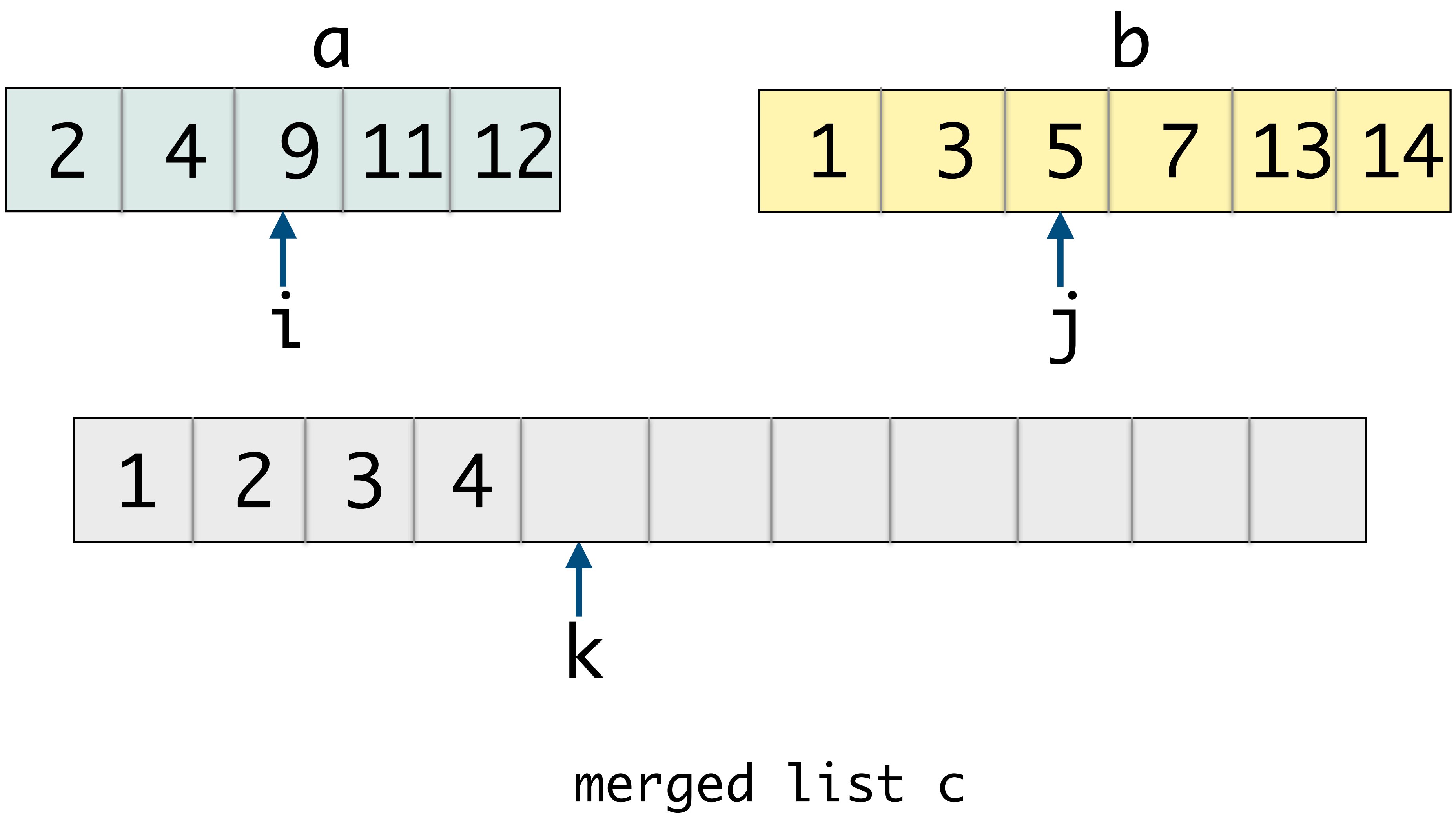

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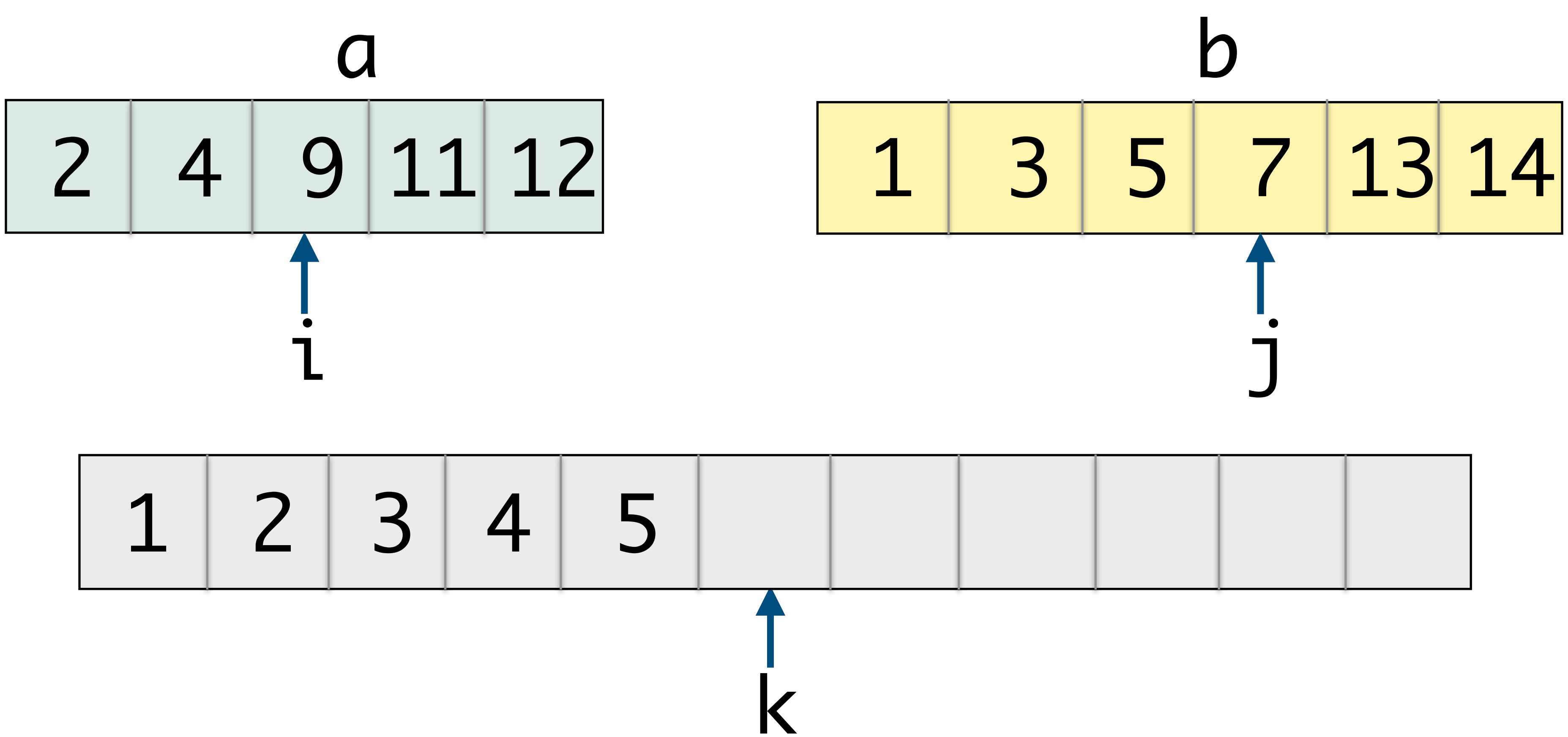

`merged list c`

Yada yada yada…

Merge Step: $\Theta(n)$

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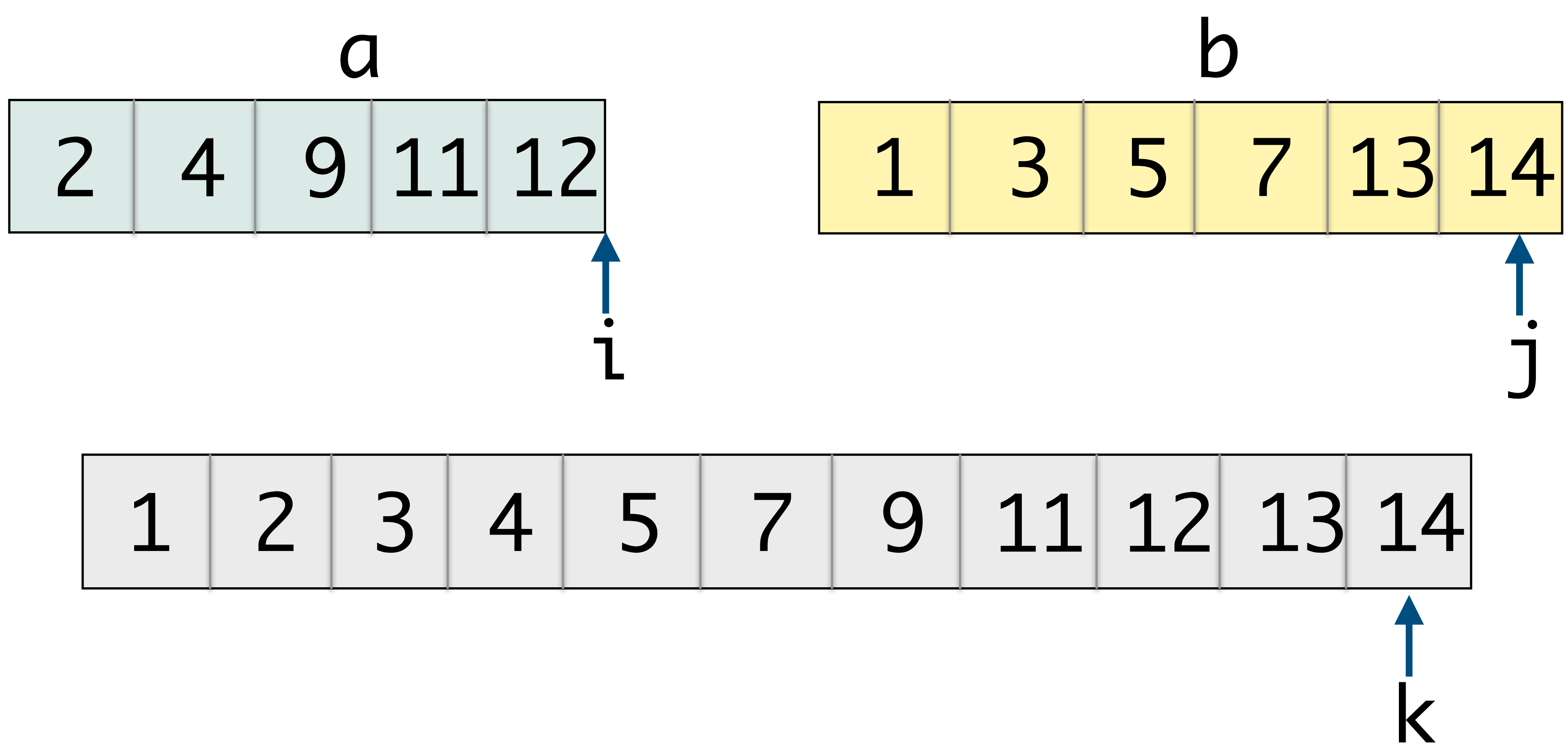


merged list c

Correctness: D&C Algorithms

- **Proving Correctness** (often follow proof by induction pattern)
  - Show base case holds
  - Assume your recursive calls return the correct solution (induction hypothesis)
  - Inductive step: crux of the proof
    - Must show that the solutions returned by the recursive calls are **"combined"** correctly

Correctness Sketch: Merge Sort

- **Claim. (Combine step.)** Merge subroutine correctly merges two sorted subarrays $A[1,\ldots i]$ and $B[1,\ldots,j]$ where $i+j=n$.
    - Will prove that for the first k iterations of the loop, correctly merges A and B (from $n=0$ to $n=k$).
- **Invariant:** Merged array is sorted after every iteration.
- Base case: $k=0$
    - Algorithm correctly merges two empty subarrays
- For inductive step, there are multiple cases, including $a_i \leq b_j$, $a_i > b_j$
    - for each case, must show that newly added element maintains sorted-ness

Analyzing Running Time

- For this topic, our main focus will be on analysis of running time
- We analyze the running time of recursive functions by:
  - **Considering the recursive calls**: both the number of calls made and the size of the inputs to each call
    - e.g., merge sort on an input of size n makes two recursive calls each on an input of size $n/2$
  - **The time spent "combining"** solutions ("non-recursive work") returned by recursive calls
    - e.g. merge step combines the sorted arrays in $\Theta(n)$ time
- Using the two, we typically write a **running time recurrence**

Running Time Recurrence

- Let $T(n)$ represent the worst-case running time of merge sort on an input of size n
- $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n)$
- **Base case:** $T(1) = 1$; often ignored
- We will ignore the floors and ceilings (turns out it doesn't matter for asymptotic bounds; we'll show this later)
- So the recurrence simplifies to:
  - $T(n) = 2T(n/2) + O(n)$
  - The answer to this ends up being $T(n) = O(n \log n)$
  - The next slides will discuss different ways to derive this

Recurrences: Unfolding

Method 1. Unfolding the recurrence

- Assume $n = 2^\ell$ (that is, $\ell = \log n$)
- Because we don't care about constant factors and are only upper-bounding, we can always choose smallest power of 2 that is greater than n. That is, $n < n' = 2^\ell < 2n$
- We can explicitly add in our constants

$$T(n) = 2T(n/2) + cn = 2T(2^{\ell-1}) + c2^\ell \text{ (change of variable, replace } n\text{)}$$

$$= 2(2T(2^{\ell-2}) + c2^{\ell-1}) + c2^\ell = 2^2T(2^{\ell-2}) + 2 \cdot c2^\ell$$

$$= 2^3T(2^{\ell-3}) + 3 \cdot c2^\ell$$

$$= \ldots$$

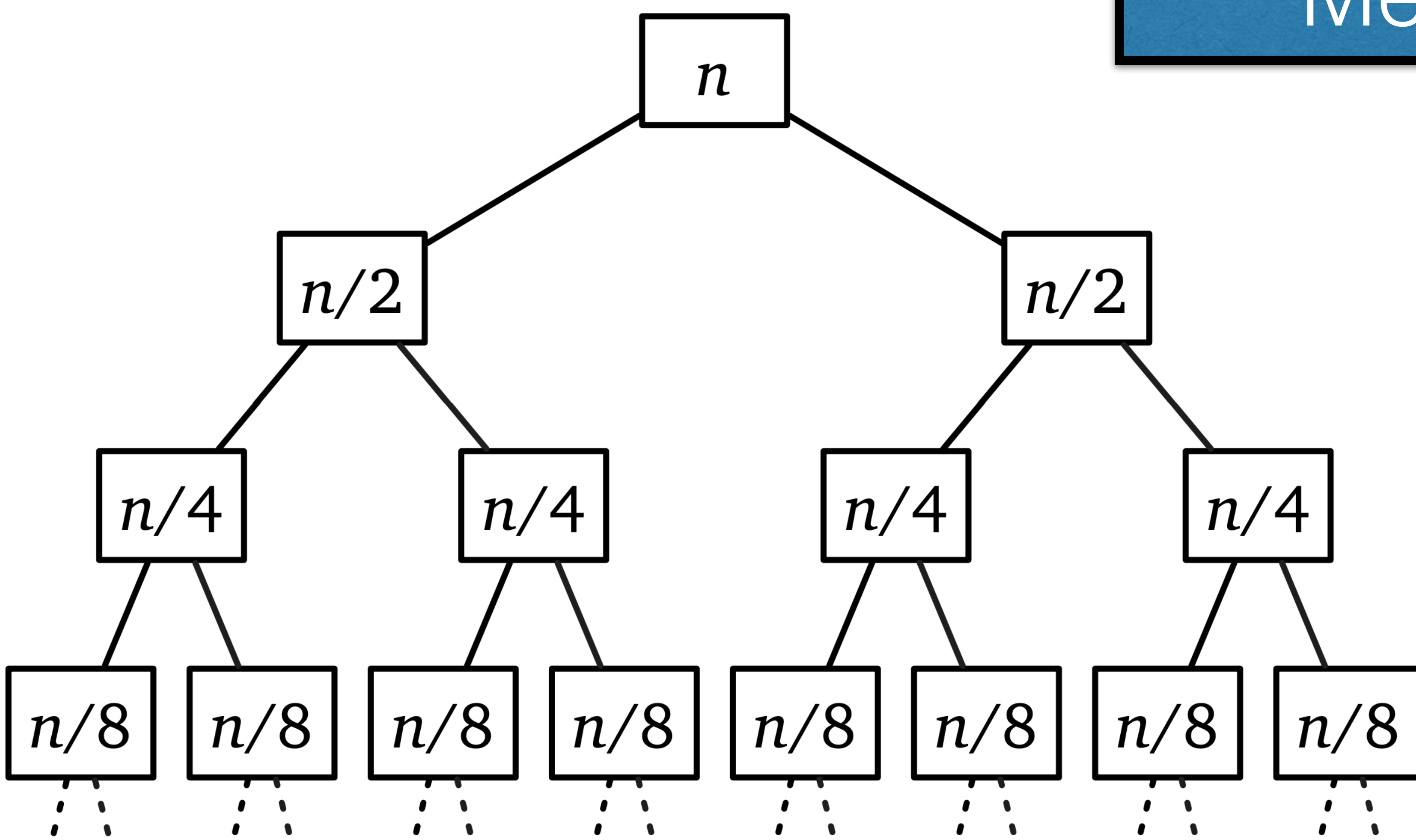
$$= 2^\ell T(2^0) + c\ell 2^\ell = O(n \log n)$$

Recurrences: Recursion Tree

Method 2. Recursion Trees

- Work done at each level $2^i \cdot (n/2^i) = n$
- Total $\log_2 n$ levels

Recommended Method!



Recurrences: Recursion Tree

- This is really a method of visualization
- Very similar to unrolling, but much easier to keep track of what's going on
- It's not (quite) a proof, but generally it is sufficient for reasoning about running times in this class
  - "Solve the recurrence" can be done by drawing the recursion tree and explaining the solution

Recurrences: Guess & Verify

Method 3. Guess and Verify

- Eyeball recurrence and make a guess
- Verify guess using induction

- More on this later…

(Anonymous) Feedback!

- What aspect(s) of the course do you like most?
- What aspect(s) of the course do you like least?
- In what CS courses that you've taken so far have you needed to spend more time than you have in this course?
- In what CS courses that you've taken so far have you needed to spend less time than you have in this course?
- Is there anything that you'd like me to know at this point in the semester?

Acknowledgments

- Some of the material in these slides are taken from
    - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
    - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)