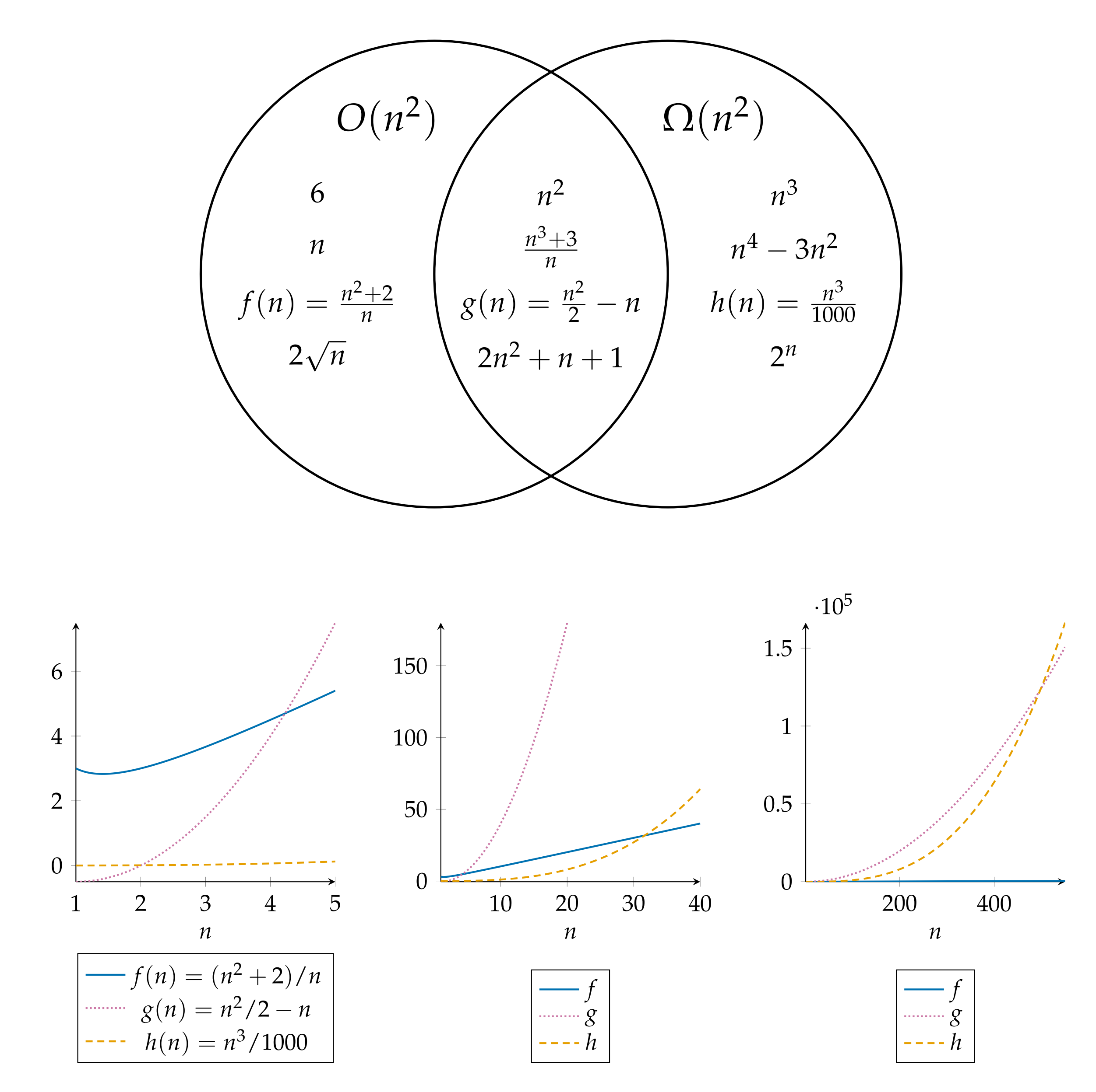
Algorithms: Introduction to Asymptotic Analysis

Model 1: Big-O and Big-Ω



Page intentionally left blank.

Critical Thinking Questions I (20 minutes)

Learning objective: Extrapolating from examples, students will develop and apply informal definitions to classify functions as $O(n^2)$, $\Omega(n^2)$, or $\Theta(n^2)$.

Process objective: Students will process information from a model to explore the meaning of big-O and big-Omega notation.

Process objective: Students will think critically to discover counterexamples and assemble evidence.

Ω is pronounced "big omega" (amusingly, "o-mega" is itself Greek for "big O", although they meant "big" in the sense of a long vowel, not uppercase).

Important note: although any previous experience you have with big-O notation may be helpful, it is **not** assumed that you remember anything in particular! When answering the following questions, as much as possible, try to rely on the information provided in Model 1 rather than on your memory.

1 **Working together**, based on the **Venn diagram** in the model, say whether each function is $O(n^2)$, $\Omega(n^2)$, or both.

(a) $\frac{n^2+2}{n}$

(b) $\frac{n^3}{1000}$

(c) $\frac{n^2}{2} - n$

(d) 2^n

For Questions 2–8, consider the functions

$$f(n) = (n^2 + 2)/n,$$
$$g(n) = n^2/2 - n, \text{and}$$
$$h(n) = n^3/1000.$$

Graphs of these functions are shown in the model (or rather, *one* graph is shown three times at different zoom levels).

2 Look at the graphs to determine which function is biggest when $2 \leq n \leq 4$.

3 The following table has four columns representing different intervals for n. For each interval, the table is supposed to show which function is smallest, which is biggest, and which is in between. A couple entries have already been filled in for you. Using

the graphs in the model, fill in the rest of the table. Note that the graphs do not quite show what happens at $n = 600$; when filling in the last column of the table, simply use your best judgment to predict what will happen.

Make sure your group agrees on the best way to fill in the table.

biggest	f			
mediumest		f		
smallest				
	$2 \le n \le 4$	$5 \le n \le 30$	$35 \le n \le 450$	$n = 600$

4 Does the same relative order continue for all $n \ge 600$, or do the functions ever change places again? Justify your answer.

5 Look at all the functions in the Venn diagram which are *both* $O(n^2)$ *and* $\Omega(n^2)$. What do they have in common?

6 Now look at the functions which are $O(n^2)$ but *not* $\Omega(n^2)$. What do they have in common?

7 Suppose we have three algorithms to solve a particular problem:

- Algorithm F takes $f(n) = (n^2 + 2)/n$ seconds to solve the problem on a particular computer when given an input of size n.
- Algorithm G takes $g(n) = n^2/2 - n$ seconds to solve the problem on a particular computer when given an input of size n.
- Algorithm H takes $h(n) = n^3/1000$ seconds to solve the problem on a particular computer when given an input of size n.

Label each of the following statements as True or False, and write a sentence or phrase explaining your reasoning.

(a) Algorithm H would be the best choice if we only ever need to solve the problem for small ($n \leq 10$) inputs.

(b) Algorithm G would be the best choice if we need to solve the problem for very large ($n \geq 10^6$) inputs.

(c) Algorithm G will take exactly $g(n) = n^2/2 - n$ seconds to solve a problem of size n on *any* computer.

(d) If we run Algorithm G on a different computer, there is some constant k such that Algorithm G will take $k \cdot g(n) = k(n^2/2 - n)$ seconds to solve a problem of size n on that computer.

(e) If we need to solve inputs of any size $n \geq 10$ in at most $n^2/4 - n/2$ seconds, we could run Algorithm G on a sufficiently fast computer.

(f) If we need to solve inputs of any size $n \geq 10$ in at most $n^2/4 - n/2$ seconds, we could run Algorithm F on a sufficiently fast computer.

(g) If we need to solve inputs of any size $n \geq 10$ in at most $n^2/4 - n/2$ seconds, we could run Algorithm H on a sufficiently fast computer.

(h) Suppose Algorithm J solves the problem for inputs of size n in some amount of time that is $O(n^2)$. In general, assuming n may be large, we would prefer Algorithm J over Algorithm H.

8 Based on the model and your answers to the previous questions, match each statement on the left with an appropriate informal definition on the right. $q(n)$ represents an arbitrary function.

We will also discuss more formal definitions in this course!

A function $q(n)$ is $O(n^2)$

A function $q(n)$ is $\Omega(n^2)$

$q(n)$ is greater than n^2 for all $n \geq 0$

Eventually, for big enough values of n, $q(n)$ grows at the same rate or more slowly than n^2

$q(n)$ grows more slowly than n^2

$q(n) \geq n^2$ for big enough values of n

The definition of $q(n)$ has n^2 in it

$q(n)$ eventually grows at a similar rate or more quickly than n^2

9 In what ways(s) is the definition to $\Omega(n^2)$ similar to and different from that of $O(n^2)$?

10 Choose two *incorrect* definitions from the previous question. For each one, write one or two sentences explaining why it is incorrect. Be sure to mention evidence from your previous answers.

Critical Thinking Questions II (10 minutes)

11 If a function is both $O(n^2)$ and $\Omega(n^2)$, we say it is $\Theta(n^2)$. For each of the below functions, say whether you think it is $\Theta(n^2)$, only $O(n^2)$, or only $\Omega(n^2)$. Justify your answers.

Θ is pronounced "big theta".

(a) $3n^2 + 2n - 10$

(b) $\dfrac{n^3 - 5}{n}$

(c) $\dfrac{n^3 - 5}{\sqrt{n}}$

(d) $(n+1)(n-2)$

(e) $n + n\sqrt{n}$

(f) $n^2 \cdot \log_2 n$

12 In your answers to Question 11, in which cases did you make use of evidence from the model (the Venn diagram or graphs) to justify your answers? In which cases did you make use of team members' previous knowledge?