Lecture 8

Homework #8: 2.2.6, 2.2.7, 2.2.9, 2.2.10. Give an NFA that accepts the language \((a \cup b)^*ab\cdot a(a \cup b)ab^*\), and then find and equivalent DFA.

Last time: introduced nondeterministic finite automata.

Now, let's show that NFAs = DFAs. [Began this last time, seeing how transitions on strings of length > 1 could be eliminated by “stretching” the automaton.]

We'll say that two finite automata M1 and M2 are equivalent iff 
\(L(M_1) = L(M_2)\).

Thm. For each NFA, there is an equivalent DFA.
Proof will proceed by construction. That is, we will show how to turn an NFA into a DFA.

To do this, we need to:
(1) eliminate transitions on e.
(2) eliminate transitions on strings of length > 1.
(3) add transitions to have actions for all symbols in a given state.
(4) eliminate multiple transitions from 1 state.

(2) is easy - just "stretch" the NFA by adding states.

Example.

\[
\begin{array}{c}
q_0 \\
q_1 \\
q_2 \\
q_3 \\
q_4
\end{array} \quad \begin{array}{c}
a,b \\
e \\
b \\
a \\
e
\end{array} \quad \begin{array}{c}
ab \\
ab \\
a \\
ab \\
a
\end{array}
\]

becomes
(of course, we'd need to prove that the "stretched" NFA is equiv to the original, but it's fairly obvious)

Basically, if \((q, u, q^1) \in \Delta\), and \(|u| > 1\), then we can write \(u = u_1 u_2 ... u_n\).
To "stretch" the NFA, we add transitions to \(\Delta\):
\[(q, u_1, p_1), (p_1, u_2, p_2), ..., (p_{n-1}, u_n, q^1).\]

Let's call the initial NFA \(M = (K, \Sigma, \Delta, s, F)\), and the stretched NFA \(M^1 = (K^1, \Sigma, \Delta^1, s^1, F^1)\).

Now, for the remainder of the proof:
We will view the NFA as follows: at **any point in time, it can be in many states at once.**

Example cont'd. On input ab, it can be in any of \(\{q0, q1, q2, q4\}\).
We can view this as a single state in a DFA.
The idea is that we're building "multi-states".

Now, what are subsequent states?
Anything that can be reached from one of these on a given input symbol.

So, if the next symbol were "a": \(\{q0, q1, p, q3\}\)

Basically, **the DFA is simulating all moves of the NFA simultaneously.**

Now let's look at transitions on e - these need to be considered specially.
Define the states that are reachable from a state \( q \) on no input:

\[
E(q) = \{ p \in K^1 : (q, e) \xrightarrow{*} (p, e) \}
\]

or

\[
E(q) = \{ p \in K^1 : (q, w) \xrightarrow{*} (p, w) \}
\]

In our Example.

\[
\begin{align*}
E(q_0) &= \{ q_0, q_1 \} \\
E(q_1) &= \{ q_1 \} \\
E(q_2) &= \{ q_2 \} \\
E(q_3) &= \{ q_3, q_4 \} \\
E(q_4) &= \{ q_4 \} \\
E(p) &= \{ p \}
\end{align*}
\]

Before giving the formal construction, let’s take these ideas and complete the construction of a DFA from the NFA above:

\[
S'' = \{ q_0, q_1 \}
\]

\[
\begin{align*}
\delta''(S'', a) &= E(q_0) \cup E(p) = \{ q_0, q_1, p \} = t_1 \\
\delta''(S'', b) &= E(q_0) \cup E(q_2) = \{ q_0, q_1, q_2 \} = t_2
\end{align*}
\]

\[
\begin{align*}
\delta''(t_1, a) &= E(q_0) \cup E(p) = \{ q_0, q_1, p \} = t_1 \\
\delta''(t_1, b) &= E(q_0) \cup E(q_2) \cup E(q_4) = \{ q_0, q_1, q_2, q_4 \} = t_3 \text{ (FINAL STATE)}
\end{align*}
\]

\[
\begin{align*}
\delta''(t_2, a) &= E(q_0) \cup E(p) \cup E(q_3) = \{ q_0, q_1, p, q_3, q_4 \} = t_4 \text{ (FINAL)} \\
\delta''(t_2, b) &= E(q_0) \cup E(q_2) = \{ q_0, q_1, q_2 \} = t_2
\end{align*}
\]

\[
\begin{align*}
\delta''(t_3, a) &= E(q_0) \cup E(p) \cup E(q_3) = \{ q_0, q_1, p, q_3, q_4 \} = t_4 \\
\delta''(t_3, b) &= E(q_0) \cup E(q_2) = \{ q_0, q_1, q_2 \} = t_2
\end{align*}
\]

\[
\begin{align*}
\delta''(t_4, a) &= E(q_0) \cup E(p) \cup E(q_3) = \{ q_0, q_1, p, q_3, q_4 \} = t_4 \\
\delta''(t_4, b) &= E(q_0) \cup E(q_2) \cup E(q_4) = \{ q_0, q_1, q_2, q_4 \} = t_3
\end{align*}
\]
Now, let's state the construction of the DFA more formally:

We construct a DFA $M'' = (K'', \Sigma, \delta'', s'', F'')$, where

$K'' = 2^K$ (power set - i.e., the set of all sets of states in $M^1$)

$s'' = E(s^1)$

$F'' = \{Q \subseteq K^1 : Q \cap F^1 \neq \emptyset\}$

$\delta''(Q, \sigma) = \cup \{E(p) : p \in K^1, (q, \sigma, p) \in \Delta^1, q \in Q\}$

(a transition records all moves on a symbol, including adjacent e-transitions)

In our Example, $s'' = \{q_0, q_1\}$

Now, we need to prove that

I. This is a deterministic FA

II. It is equivalent to $M^1$

I. This part is easy - by our definition of $\delta$, $M''$ is deterministic.

II. Claim: for any $w \in \Sigma^*$, and $q, p \in K^1$

$$(q, w) \xrightarrow{\delta_{M^1}} (p, e) \text{ iff } (E(q), w) \xrightarrow{\delta_{M''}} (P, e), \quad p \in P.$$  

We will prove this by induction on the length of $w$: 
**Basis.** \( |w| = 0 \), so \( w = e \).

so we need to show
\[
(q,e) \vdash^{*_{M^1}} (p,e) \iff (E(q),e) \vdash^{*_{M^1}} (p,e), \quad p \in P.
\]

\((\Leftarrow)\) Since \( M'' \) is deterministic, \( E(q) = P \), and \( p \in P \),
so
\[
(q,e) \vdash^{*_{M^1}} (p,e), \text{ by the definition of } E(q).
\]

\((\Rightarrow)\) if \((q,e) \vdash^{*_{M^1}} (p,e)\), then \( p \in E(q) \), according to the
definition of \( E \), but then
\[
(E(q),e) \vdash^{*_{M^1}} (p,e), \quad p \in P.
\]

**Induction Step.** Assume that the claim is true for all \( w, |w| \leq k, k \geq 0 \).

Show that it also holds for \( w, |w| = k + 1 \).

Let \( w = va, v \in \Sigma^*, a \in \Sigma \).

\((\Rightarrow)\) Suppose that \((q,w) \vdash^{*_{M^1}} (p,e)\), which we can rewrite as
\[
(q,va) \vdash^{*_{M^1}} (p,e)
\]
\[
= (q, va) \vdash^{*_{M^1}} (r^1, a) \vdash^{M^1} (r^2, e) \vdash^{*_{M^1}} (p,e)
\]

Now, what can we say about the pieces of this computation?

First, rather than considering \((q, va) \vdash^{*_{M^1}} (r^1, a)\), let's think
about \((q, v) \vdash^{*_{M^1}} (r^1, e)\). The induction hypothesis tells us
that
\[
(E(q),v) \vdash^{*_{M''}} (R^1,e), \quad r^1 \in R^1.
\]

Second, since \((r^1, a) \vdash^{M^1} (r^2, e)\)
\[
(r^1, a, r^2) \text{ is in } \Delta^1
\]
so \( E(r^2) \subseteq \delta''(R^1, a) \), by definition.

Third, since \((r^2, e) \vdash^{*_{M^1}} (p,e)\), \( p \in E(r^2) \).

**Putting these all together:**
\[
p \in E(r^2) \subseteq \delta''(R^1, a), \text{ so } (R^1,a) \vdash (P,e), \quad p \in P.
\]
So

\[(E(q), va) \rightarrow^{*_{M'}} (R^1, a) \rightarrow (P, e), \quad p \in P.\]

so

\[(E(q), va) \rightarrow^{*_{M'}} (P, e), \quad p \in P.\]

The other direction is easier - we won't do it in class.